Méthodes intégrales pour l'évaluation en champ proche

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ENSTA(2009-2012)
32 Blvd Victor, Paris

MA102
MA201
MA206
MAE21
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Doctorat (2012-2015)
921 Blvd des Maréchaux, Palaiseau

La famille "changement de signe"
(K. Ramdani, C.M. Zwölf, L. Chesnel, C. Carvalho, M. Rihani, F. Chaaban, …)

JO des Poètes 2024 - Relais 4x60

The close evaluation problem

Goal: accurately evaluate the solution near the boundary.

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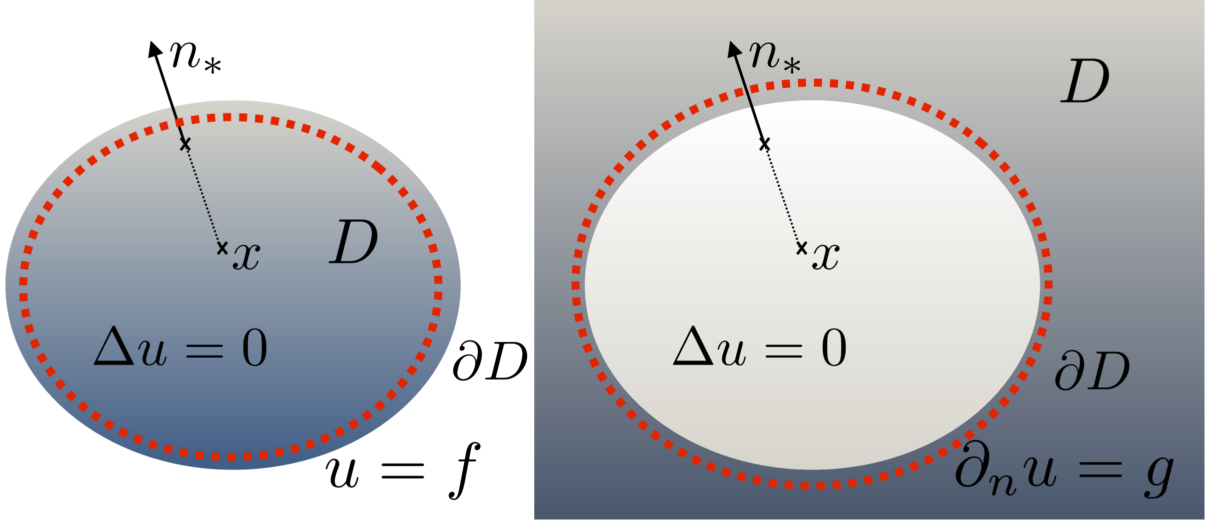
Examples:

Laplace problems



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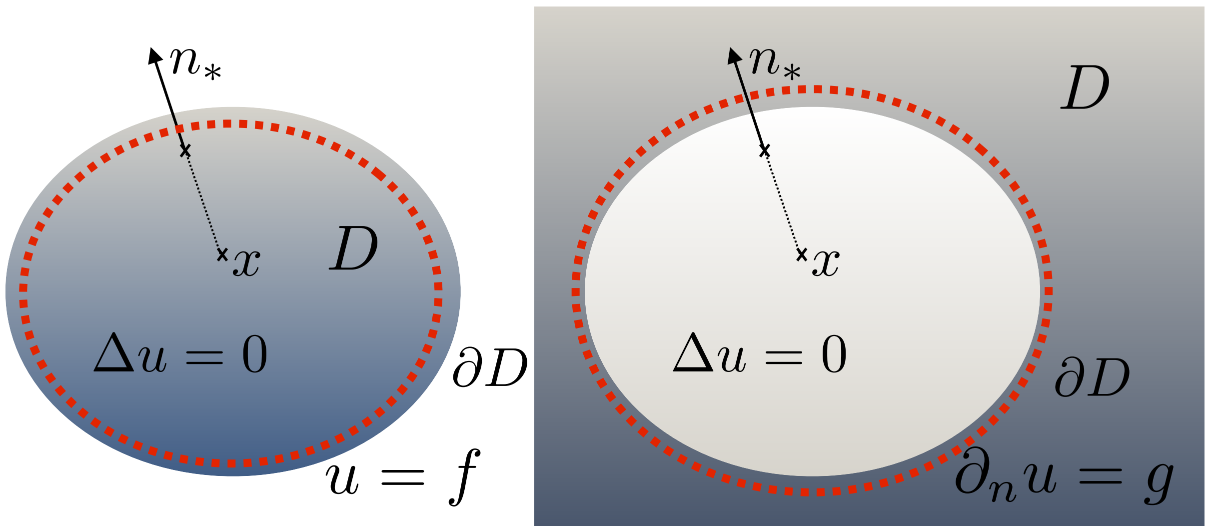
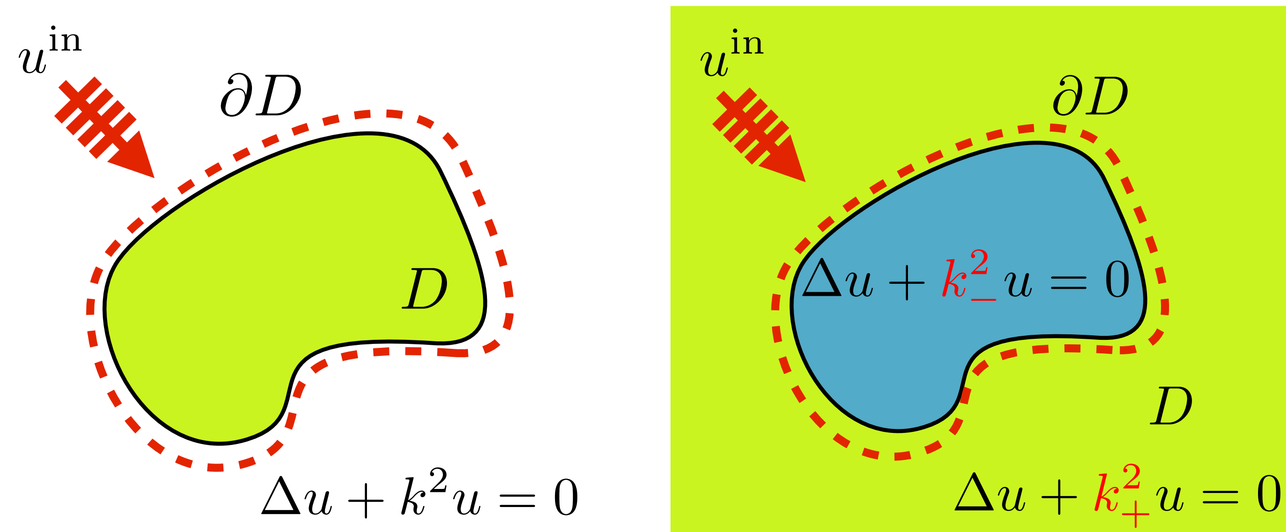


Scattering problems



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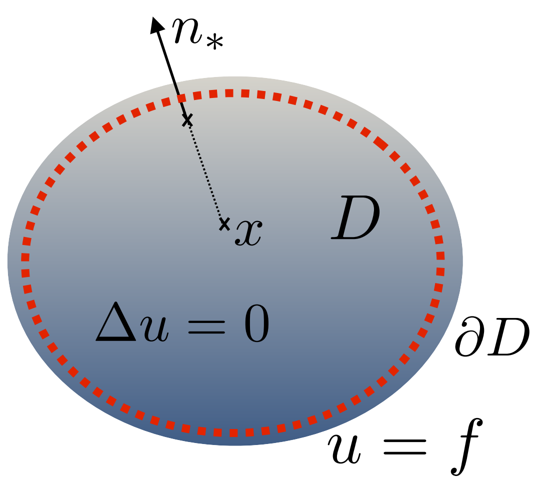

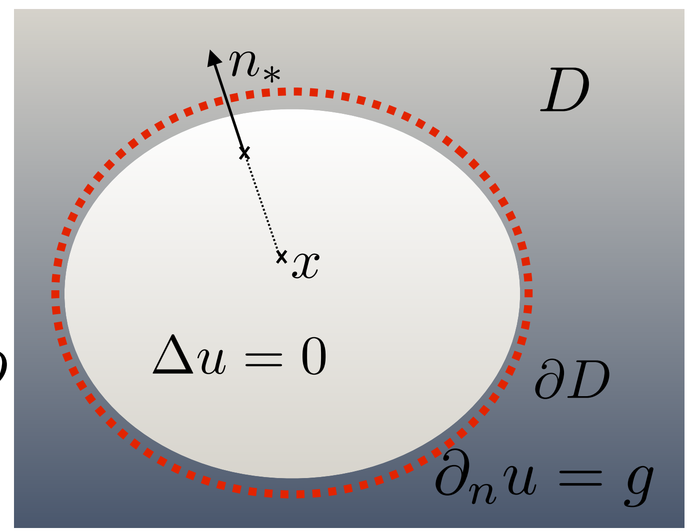

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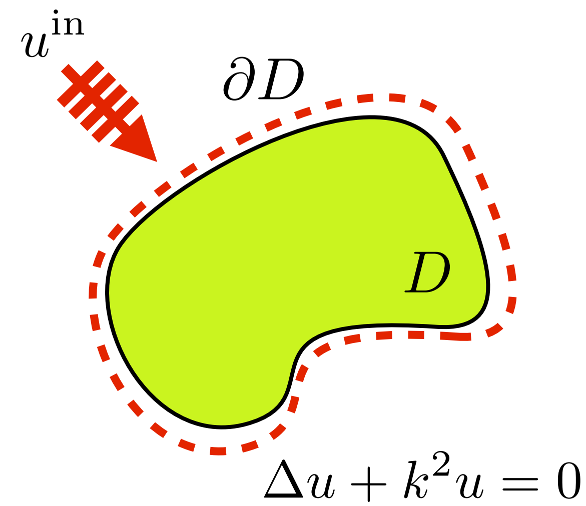

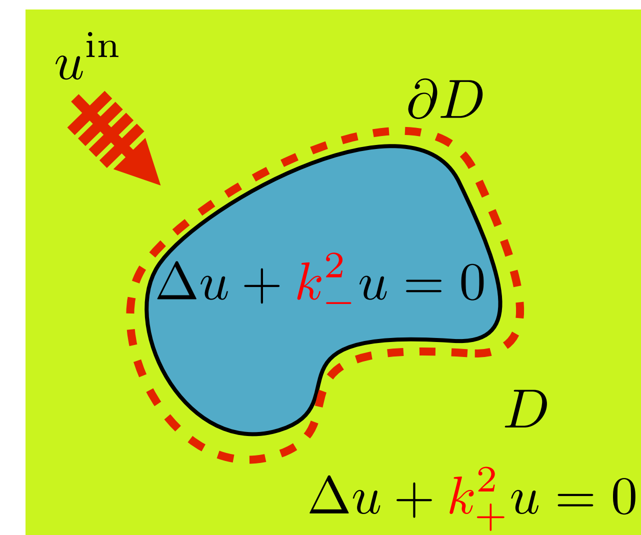

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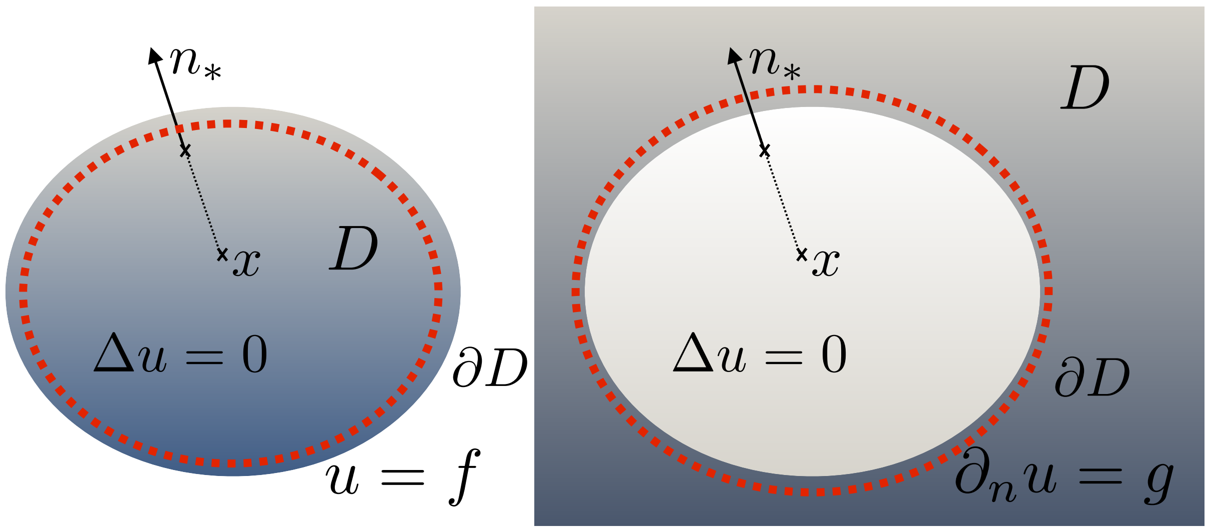

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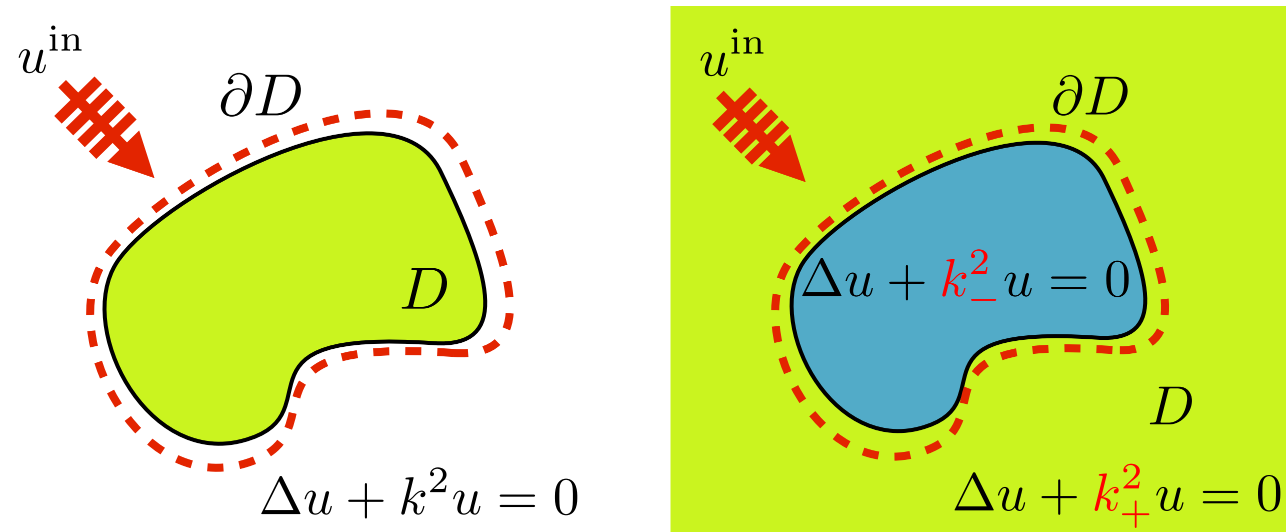

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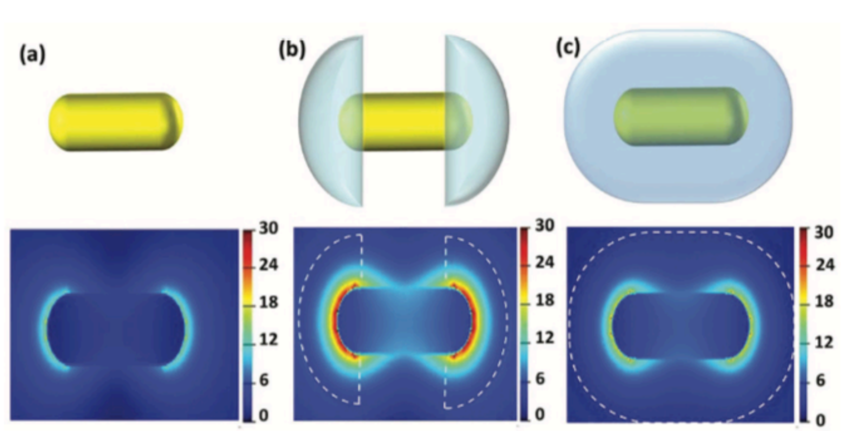
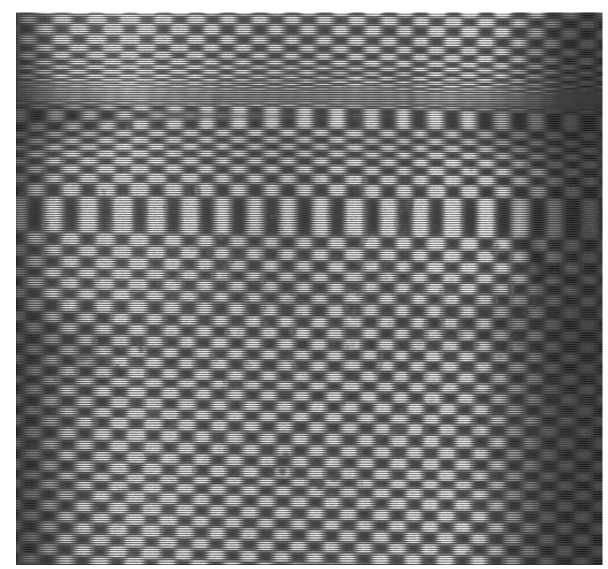


Zhu et al., Nanoscale (2020)



JFL Lab.

An example in 2D

$$\Delta u_+ + k_+^2 u_+ = 0 \quad \mathbb{R}^d \setminus \overline{D}, \quad d = 2, 3$$

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$$u_+ = u_- \quad \partial D$$

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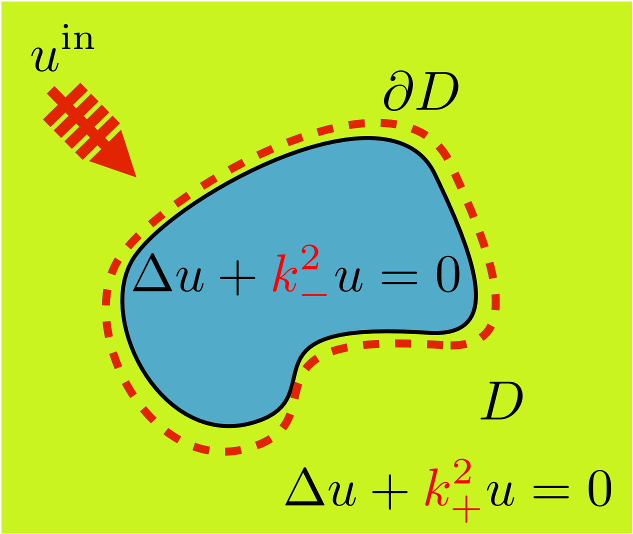
$+$ Radiation condition at ∞



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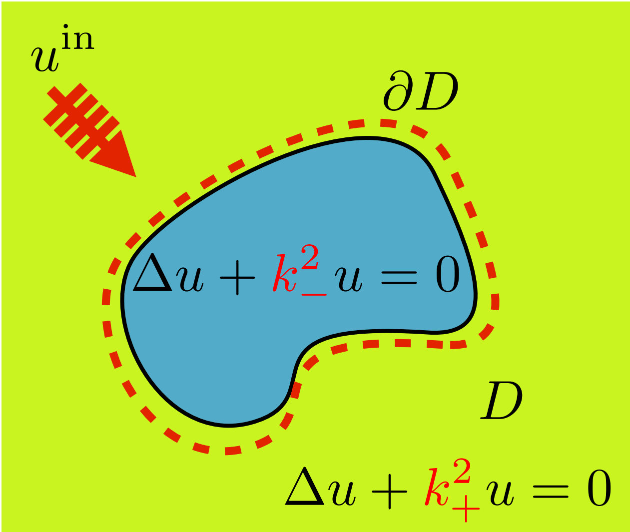

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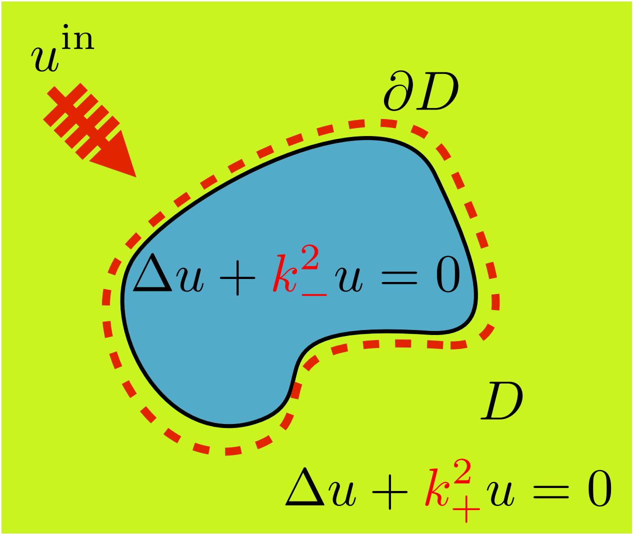

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$$u_+(x) = u^{\text{inc}}(x) + \int_{\partial D} \partial_n G^+(x,y) u_+ \, d\gamma_y - \int_{\partial D} G^+(x,y) \partial_n u_+ \, d\gamma_y$$

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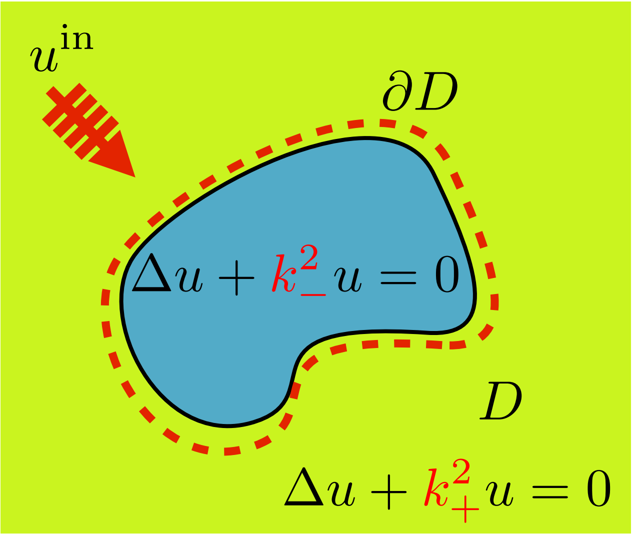

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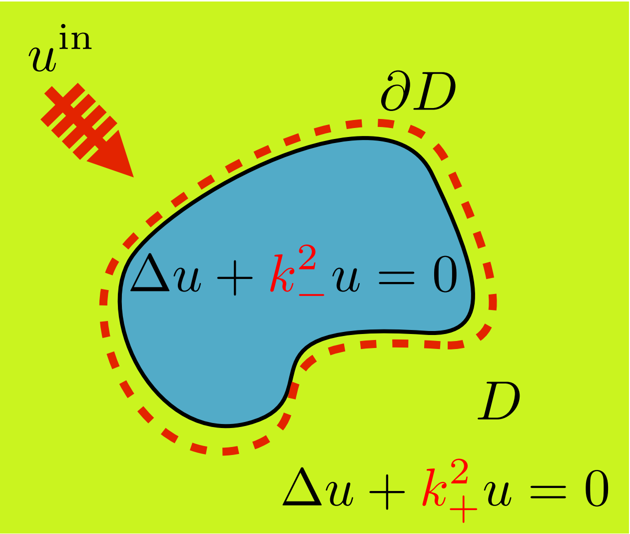

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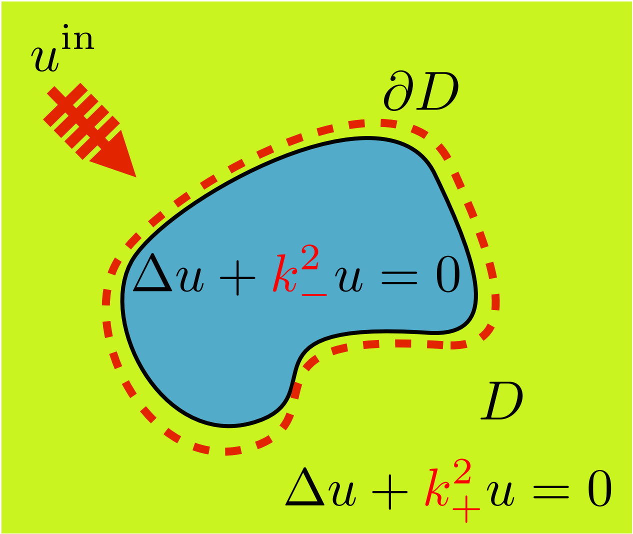

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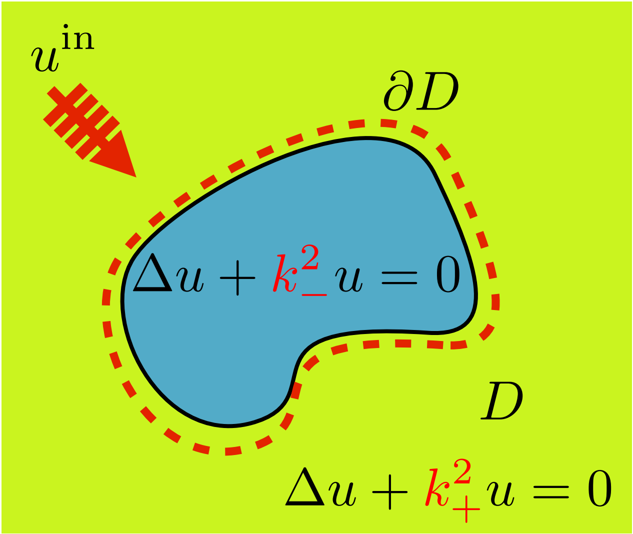

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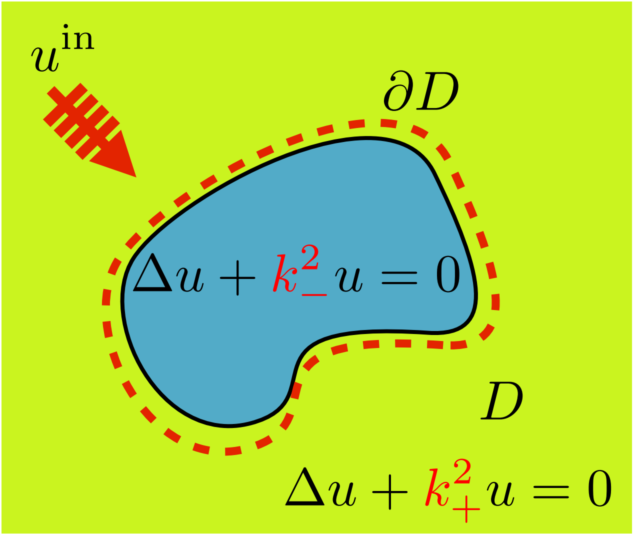

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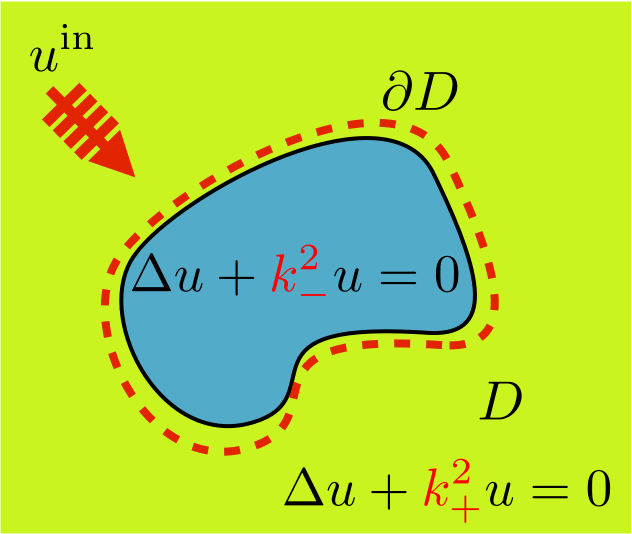

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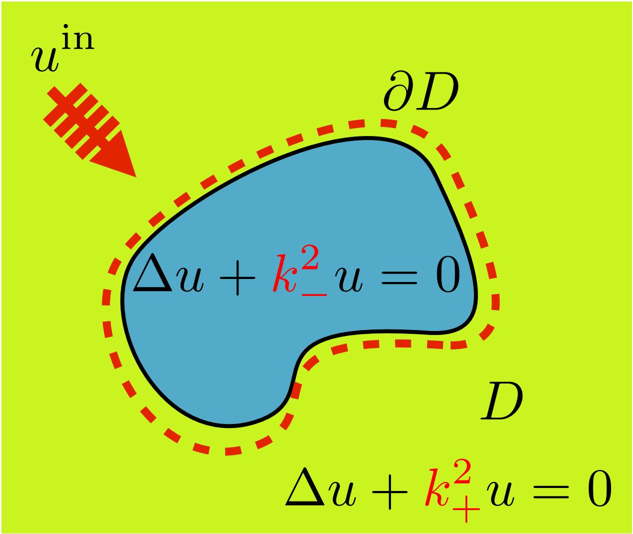

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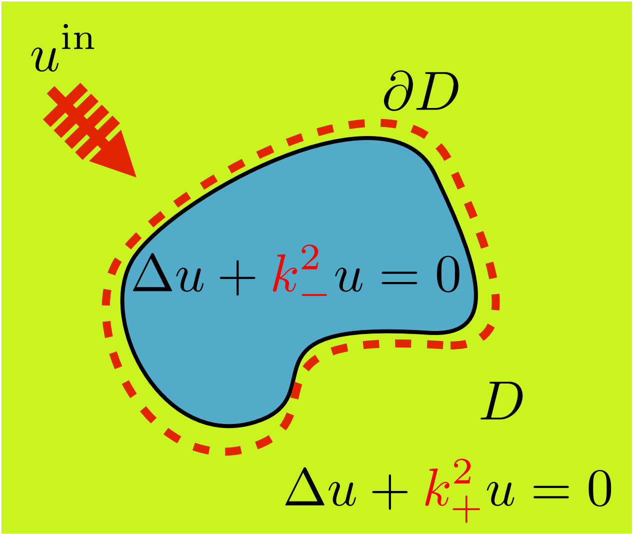

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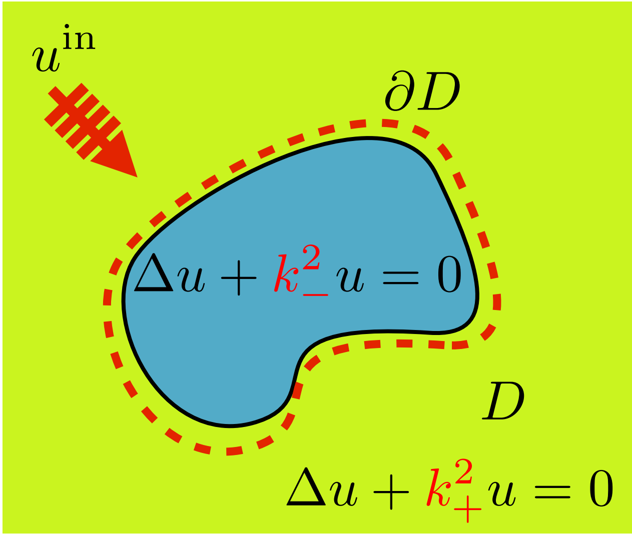

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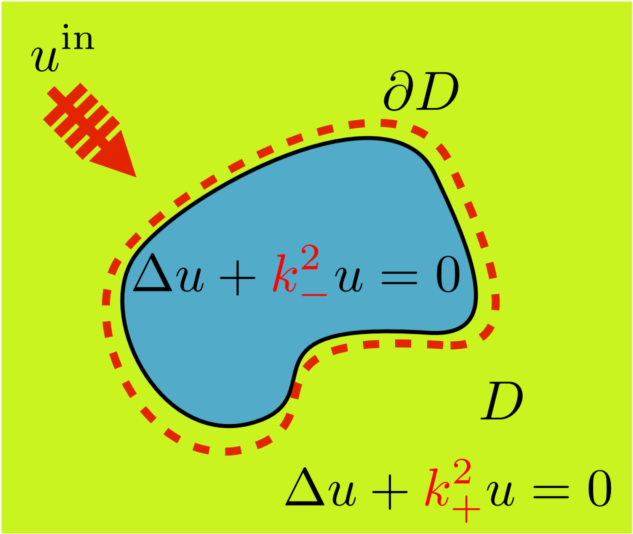

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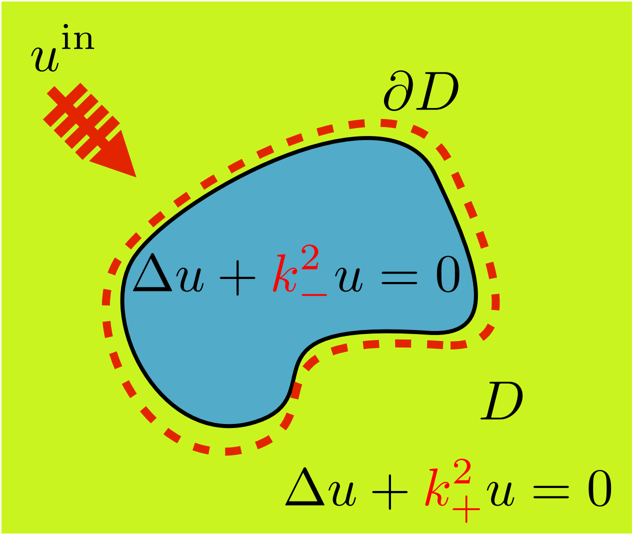

Use a quadrature rule (Nyström method) to evaluate the BIE.

$$u_{-}(x) = -\int_{\partial D} \partial_n G^{-}(x,y) u_{+} \, d\gamma_y + \frac{\sigma^{+}}{\sigma^{-}} \int_{\partial D} G^{-}(x,y) \partial_n u_{+} \, d\gamma_y$$

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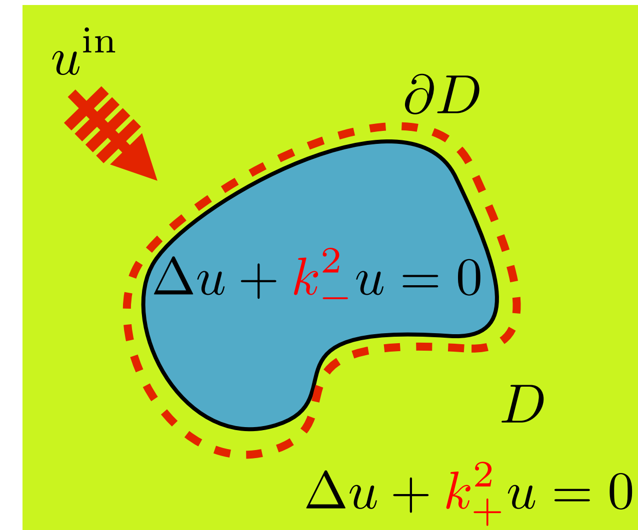

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Use the same quadrature rule for the layer potentials.

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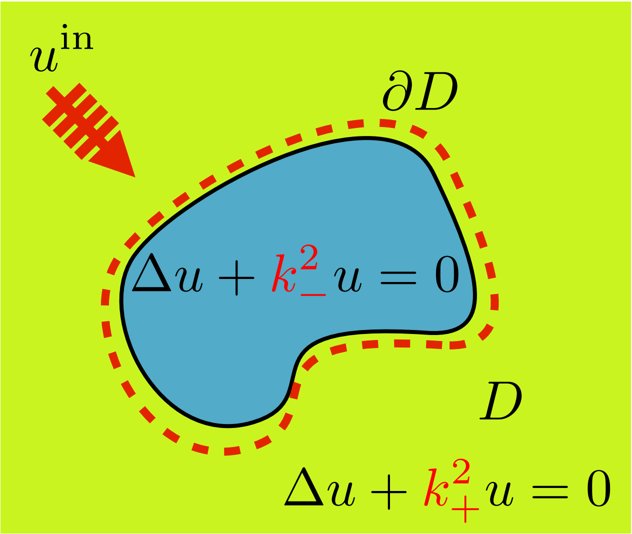

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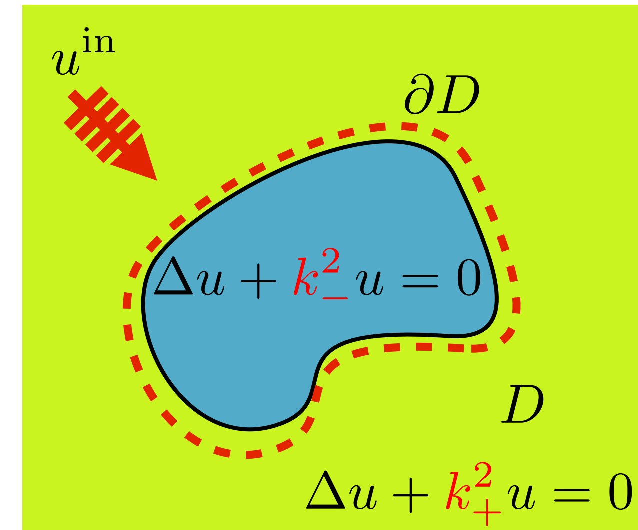

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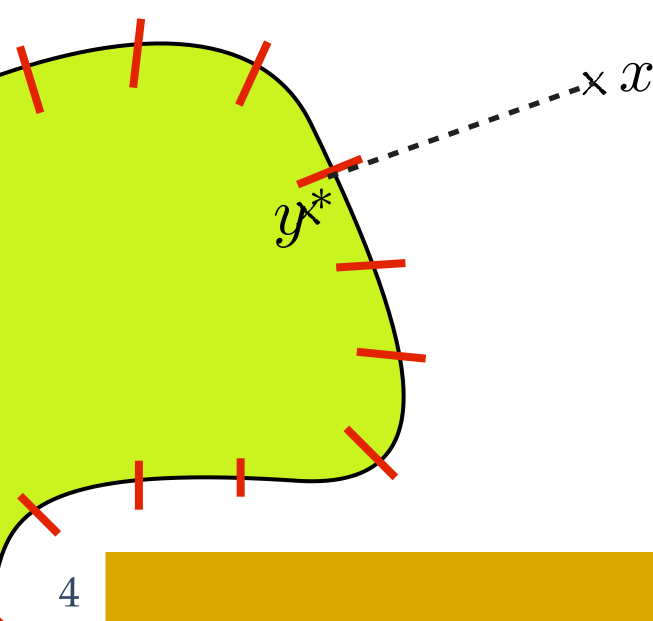

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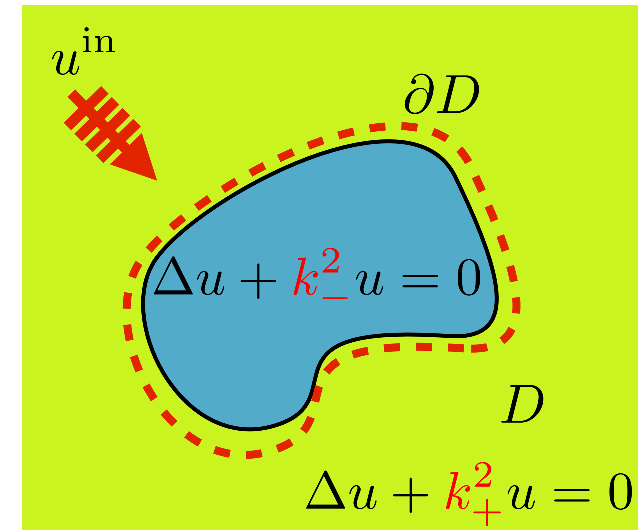

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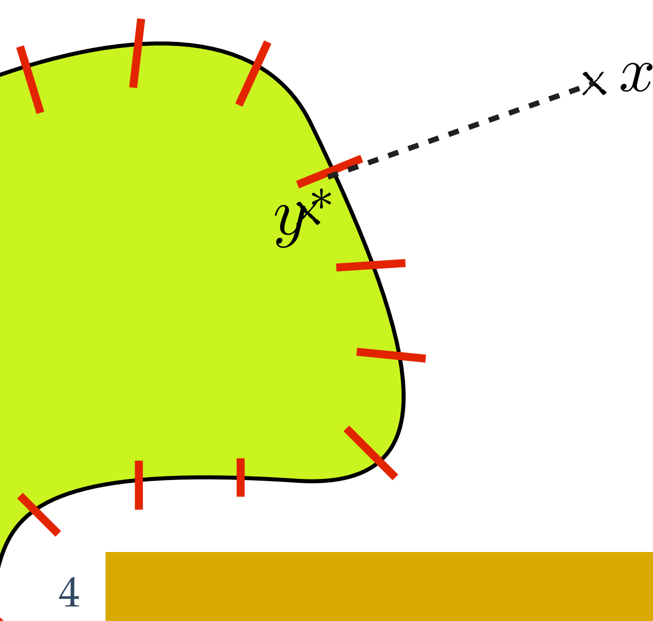

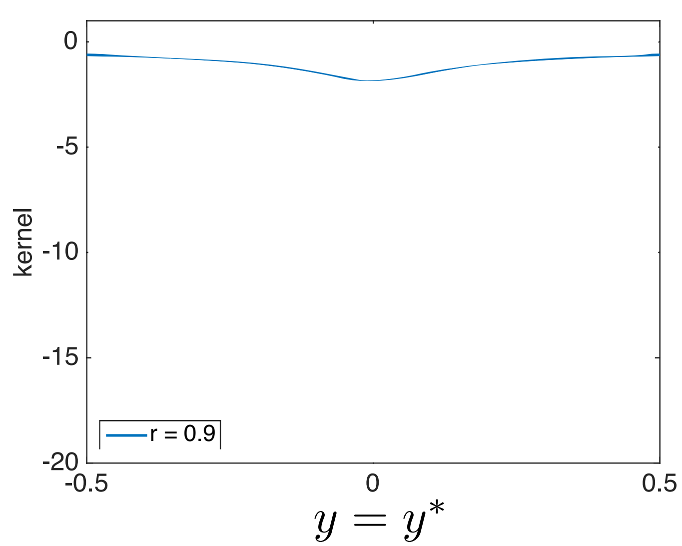

Discretization

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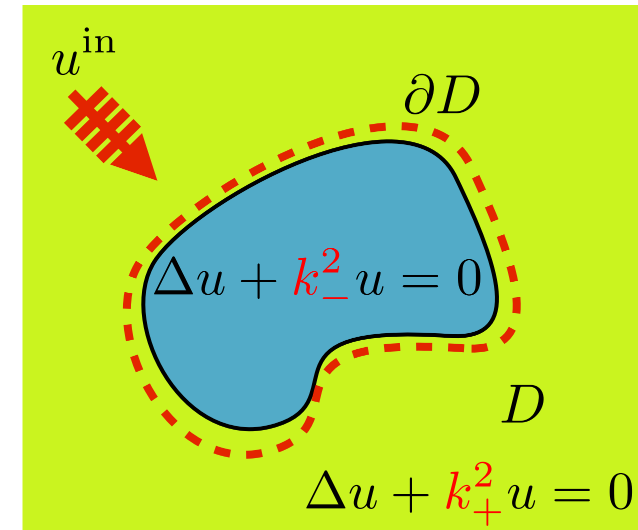

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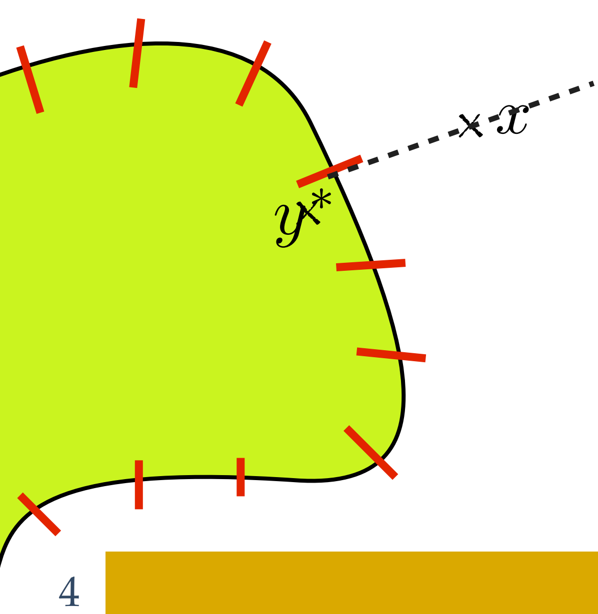

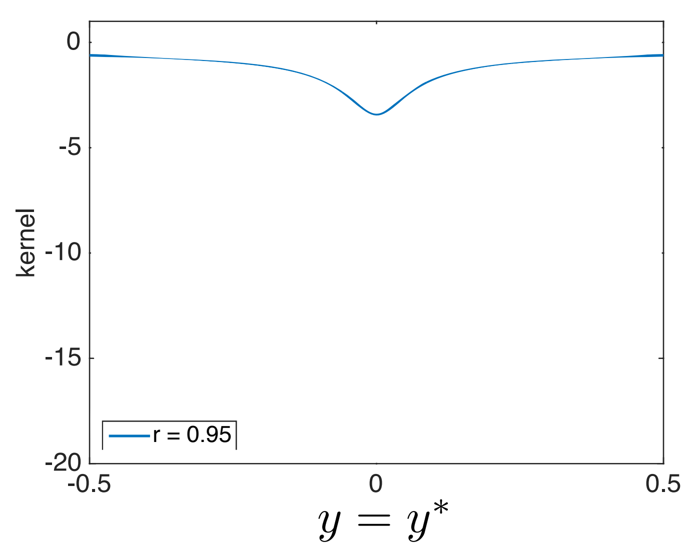

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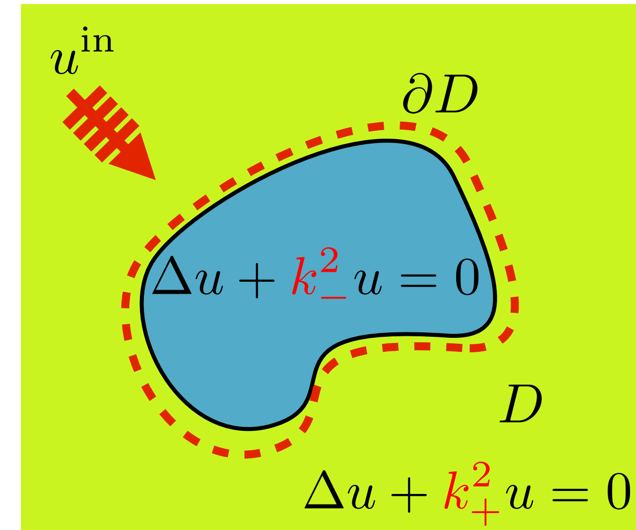

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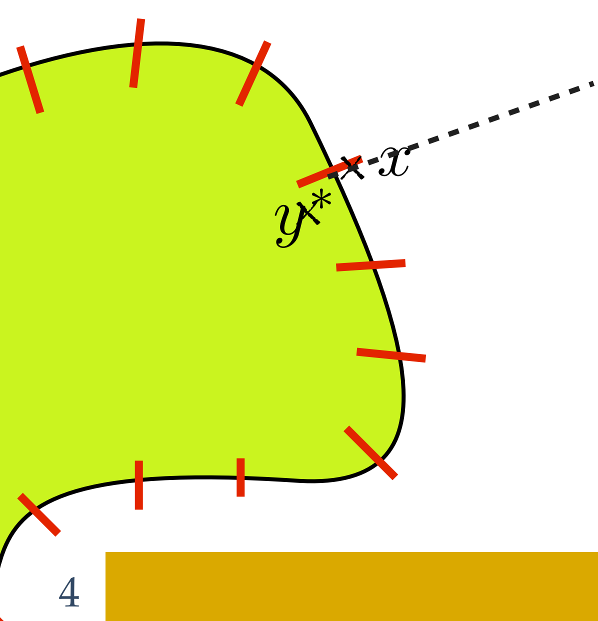

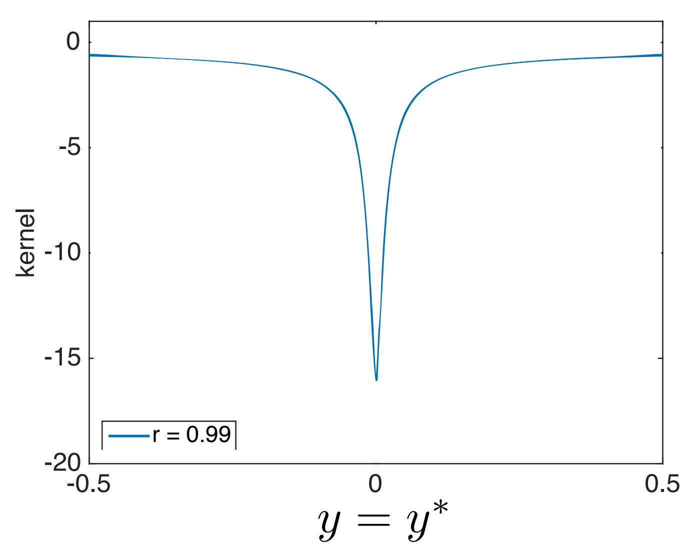

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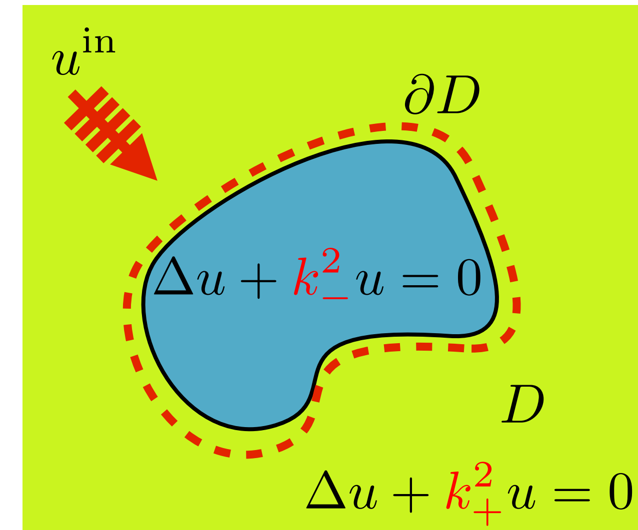

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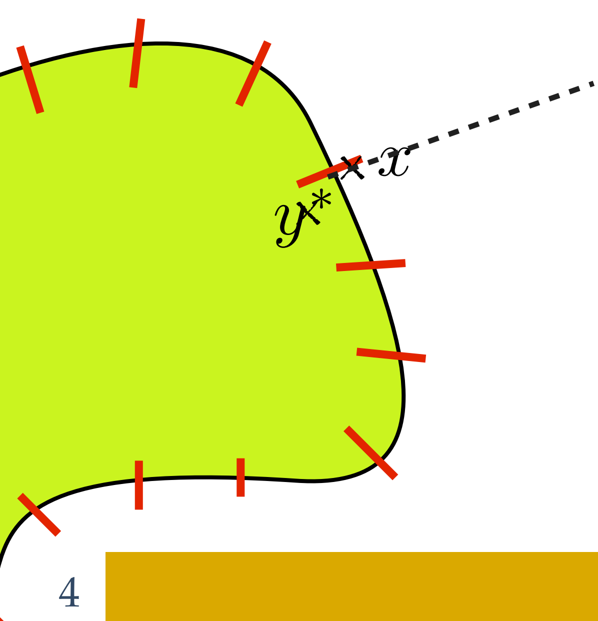

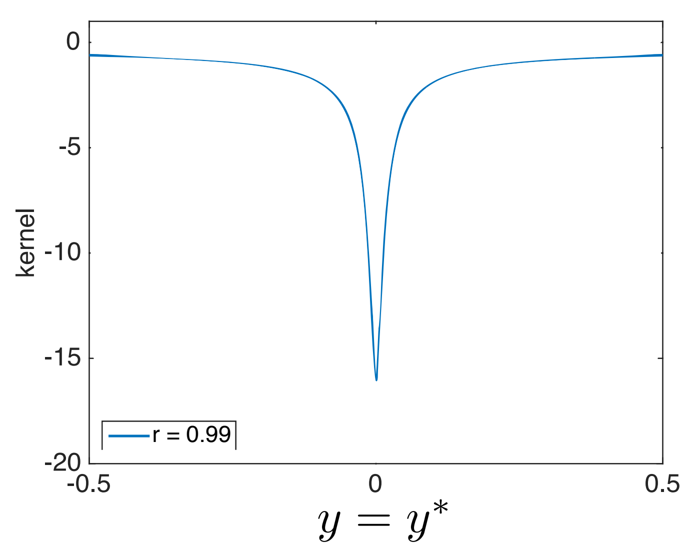

Nearly singular integral

Discretization

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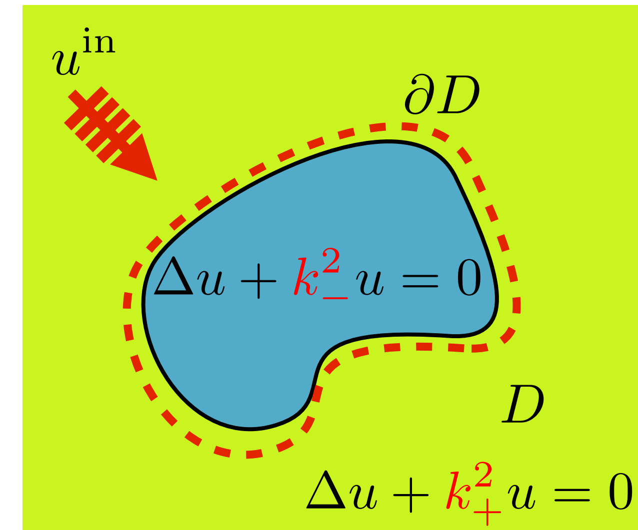

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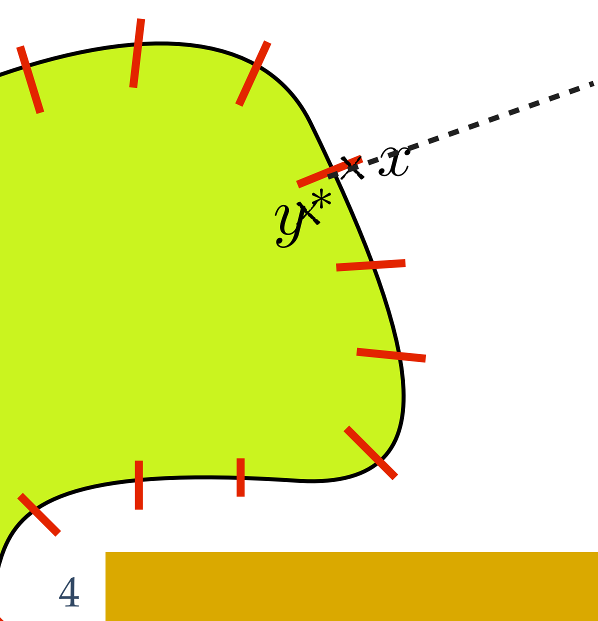

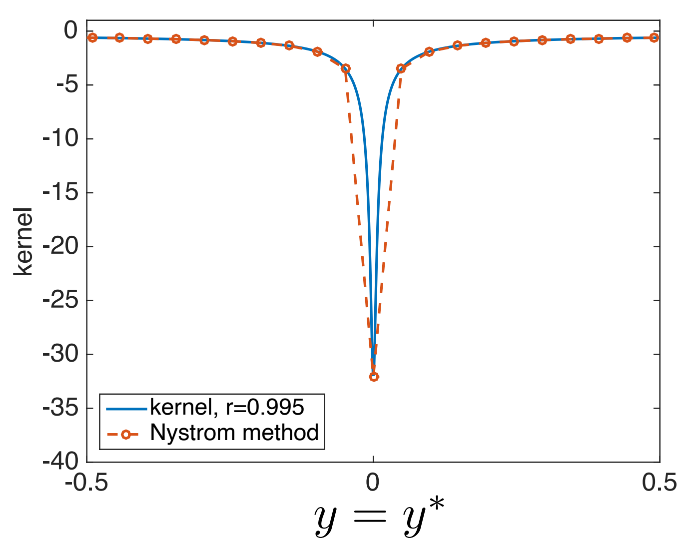

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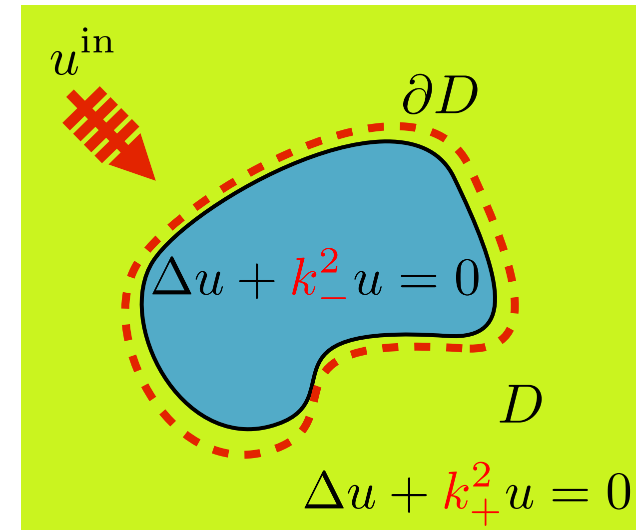

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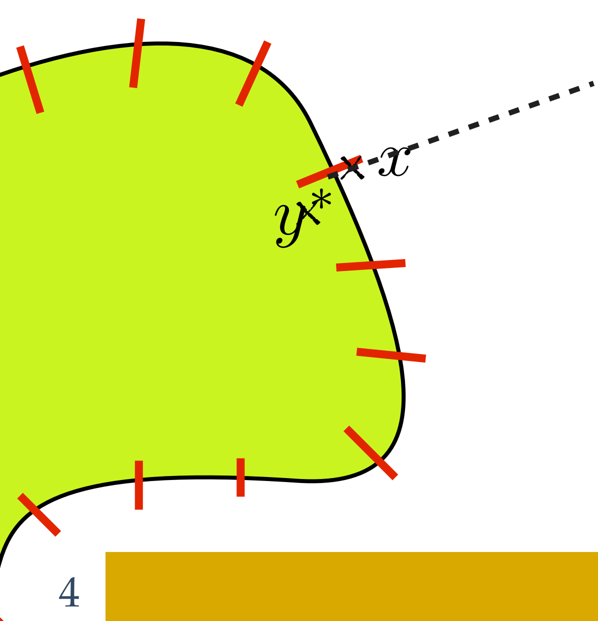
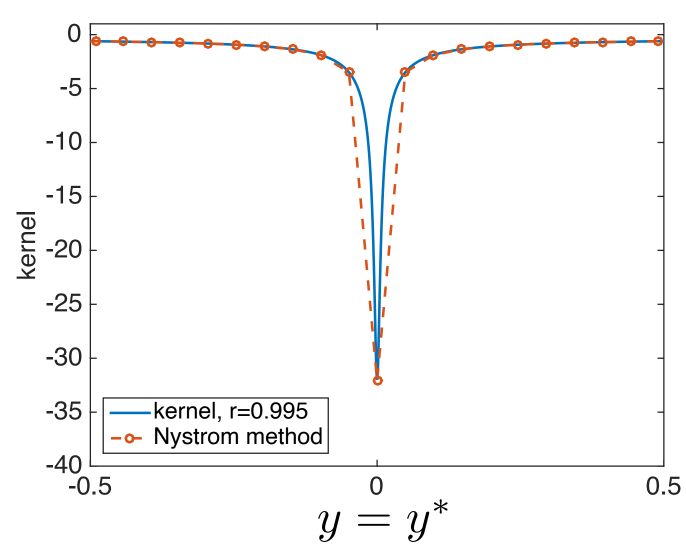
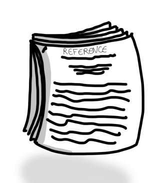




Nearly singular integral

For a fixed number of quadrature points, O(1) error.



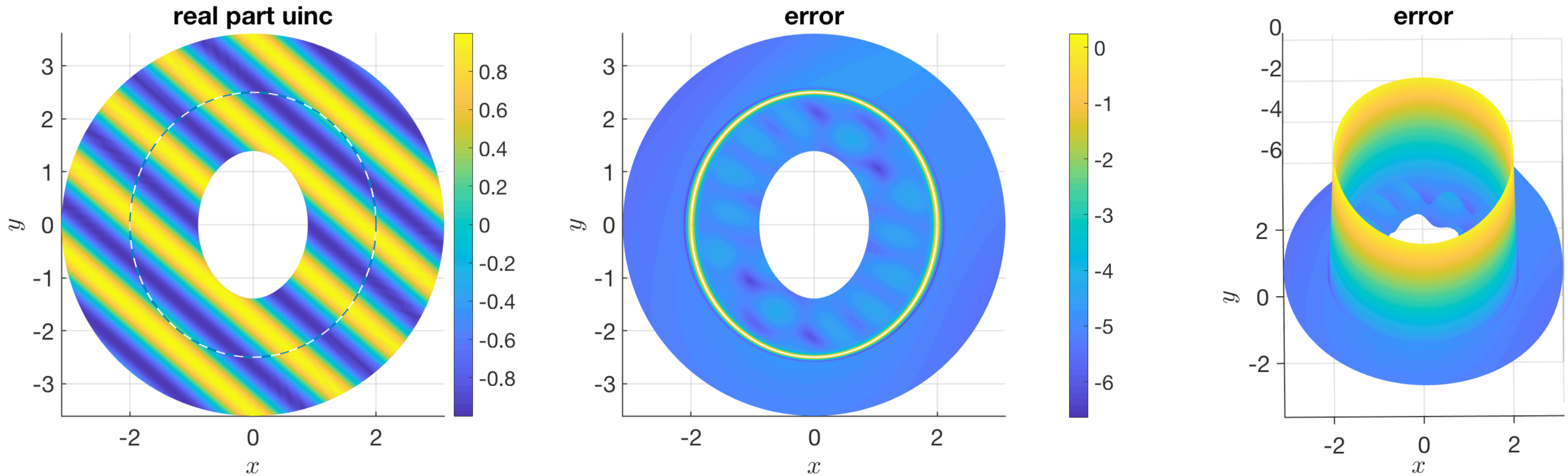
Barnett (2014).

Example

Example of a plane wave in an homogeneous domain (elliptic obstacle).

$k_-^2 = k_+^2 = 5$

Use 128 points for the quadrature.



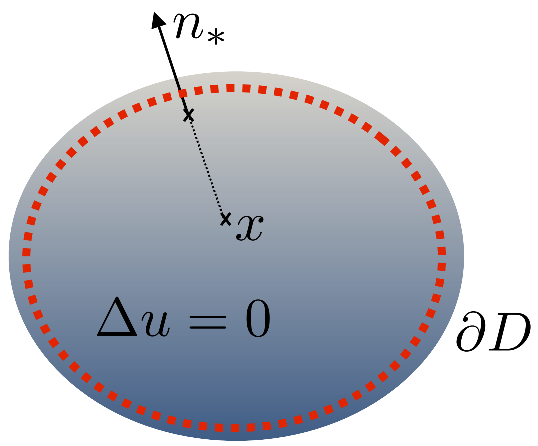
Real part solution

Log plot of the error (2D view, 3D view)

An example in 3D

Interior Dirichlet Laplace problem

$$\Delta u = 0 \quad \text{in } D, \quad u = f \quad \text{on } \partial D$$



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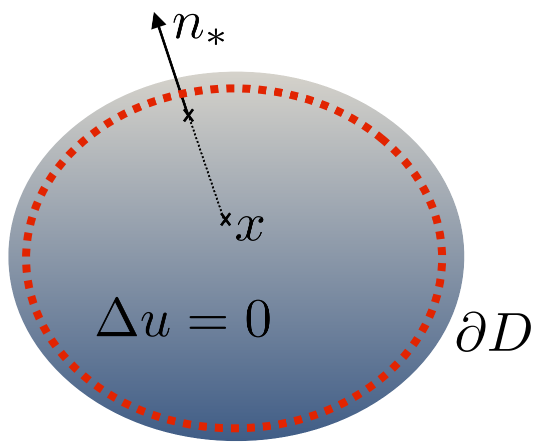

We evaluate the solution via the representation formula

$$u(x) = -\int_{\partial D} \partial_{n_y} G(x,y) u(y) d\sigma_y + \int_{\partial D} G(x,y) \partial_n u(y) d\sigma_y$$

G: fundamental solution $\quad G(x,y) = \dfrac{1}{4\pi |x-y|}$

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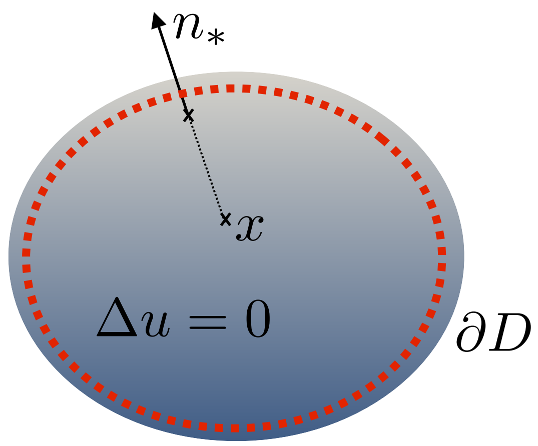

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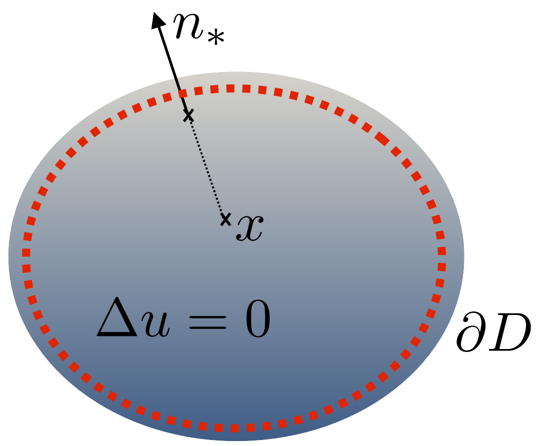

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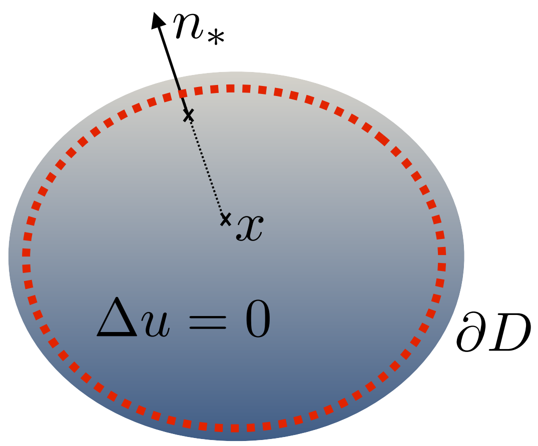

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Using parameterization

$$y = y(s,t),\ s \in [0,\pi],\ t \in [-\pi,\pi]$$

$$I(y^*) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^{\pi} F(s,t)\, \mathrm{d}s\mathrm{d}t$$

Discretization

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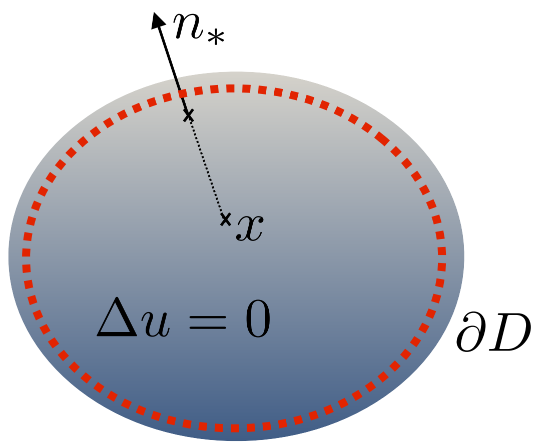

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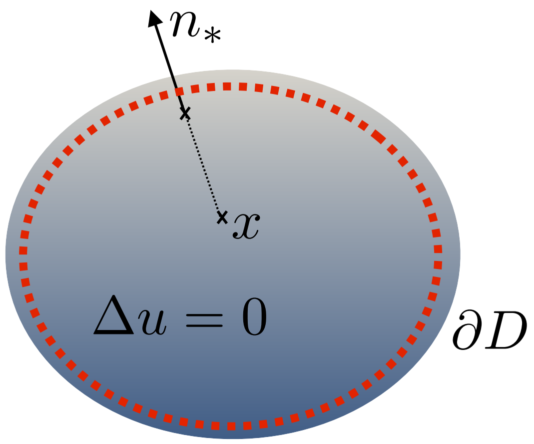
with weights w_i

2N Periodic Trapezoid rule $t_j = -\pi + \pi \frac{j-1}{N},\ j = 1, \ldots, 2N$



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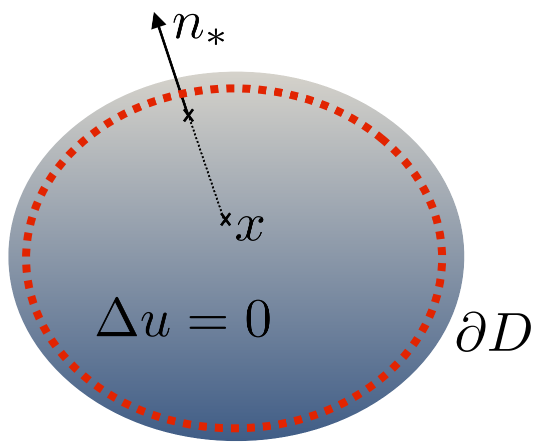

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Product Gaussian Quadrature rule Atkinson (1982)

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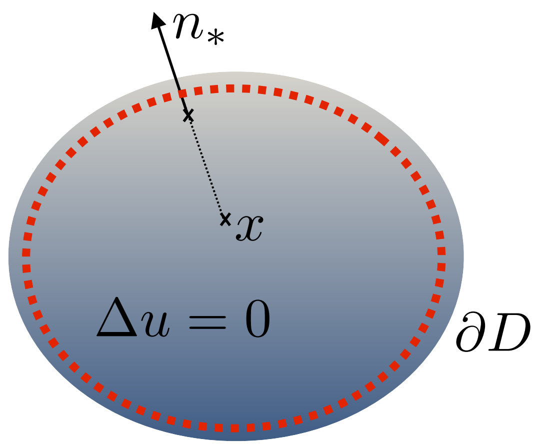

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Quadrature rules for nearly-singular integrals:

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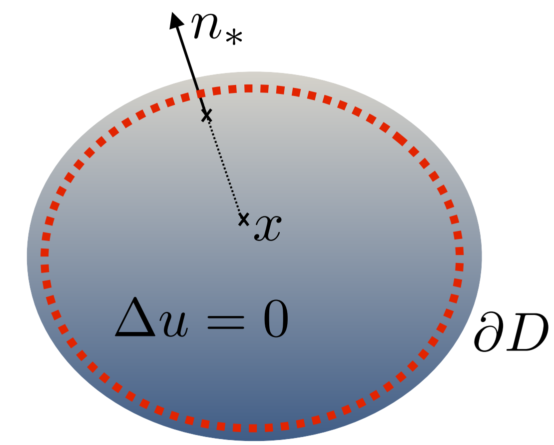

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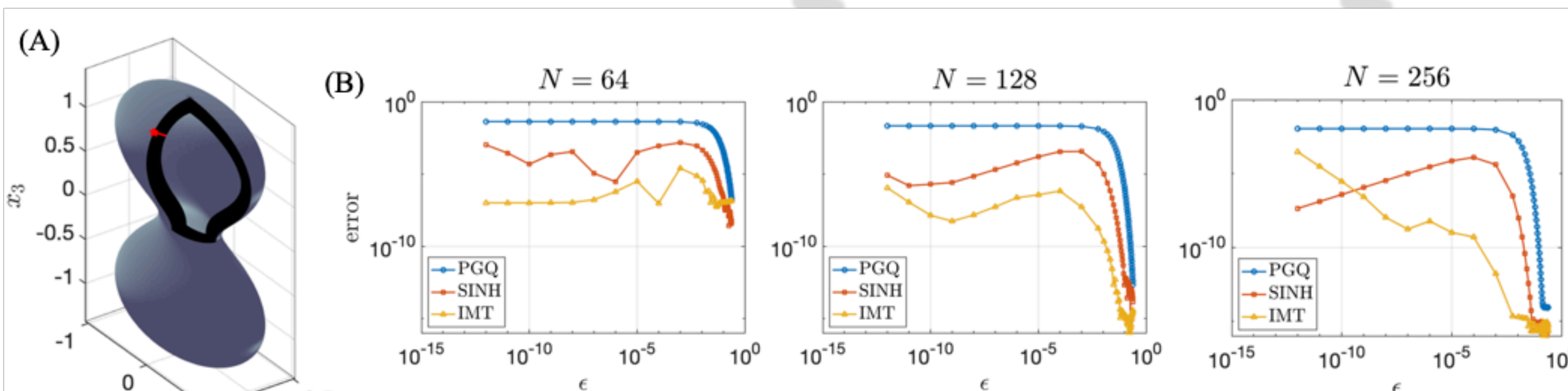
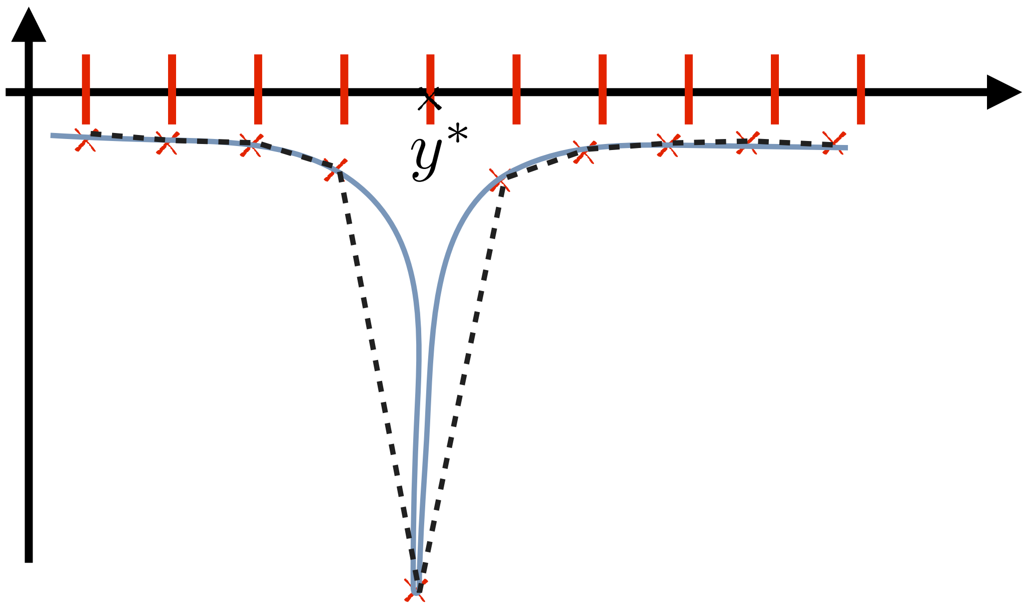


$$u(x) = -D[u_{\mathrm{ex}}](x) + S[\partial_n u_{\mathrm{ex}}](x) \qquad u_{\mathrm{ex}}(x_1,x_2,x_3) = e^{x_3}(\sin x_1 + \sin x_2)$$

How to address this error ?



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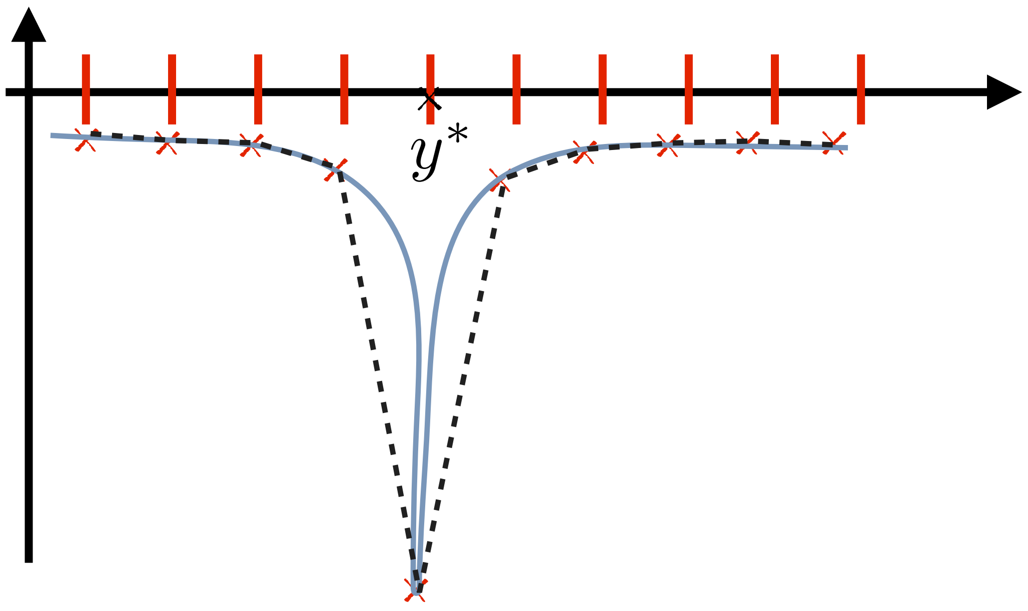

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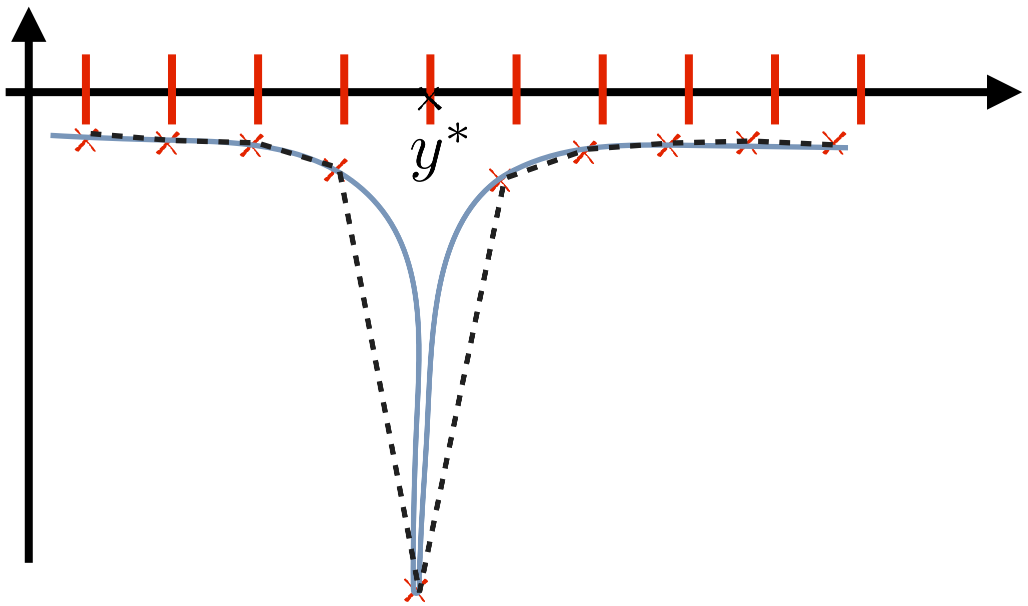

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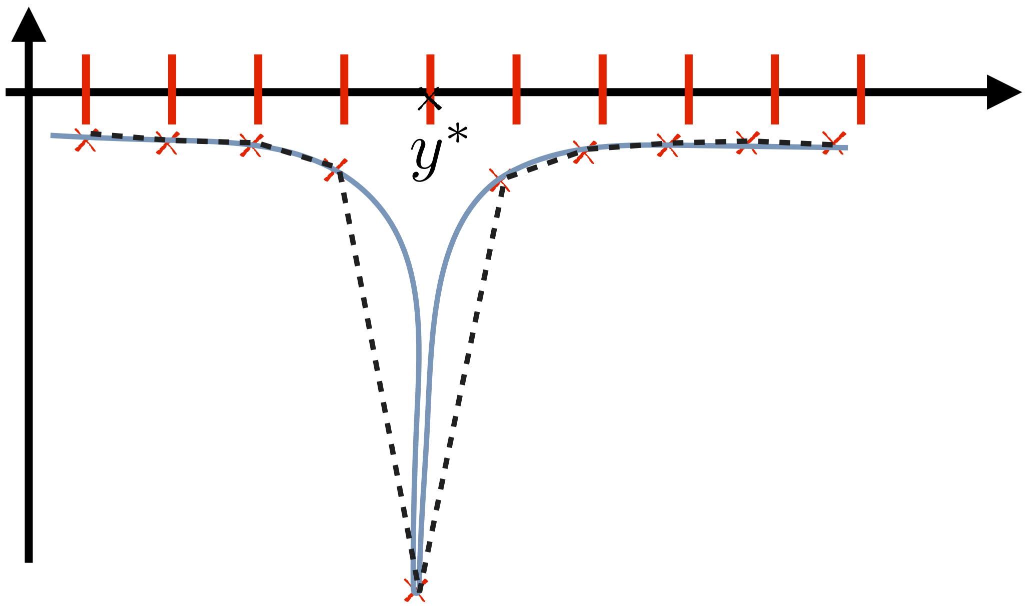

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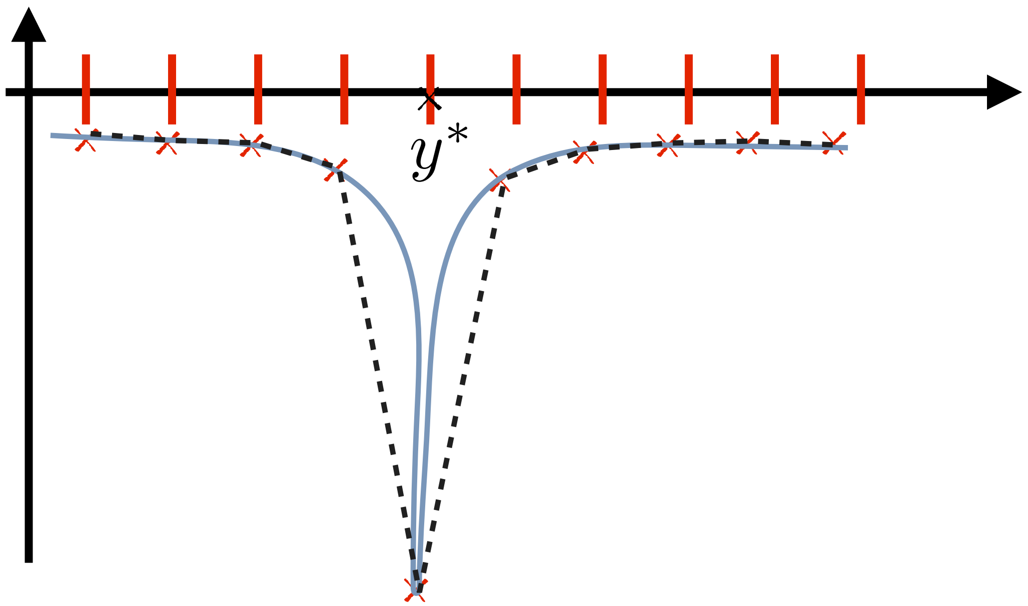

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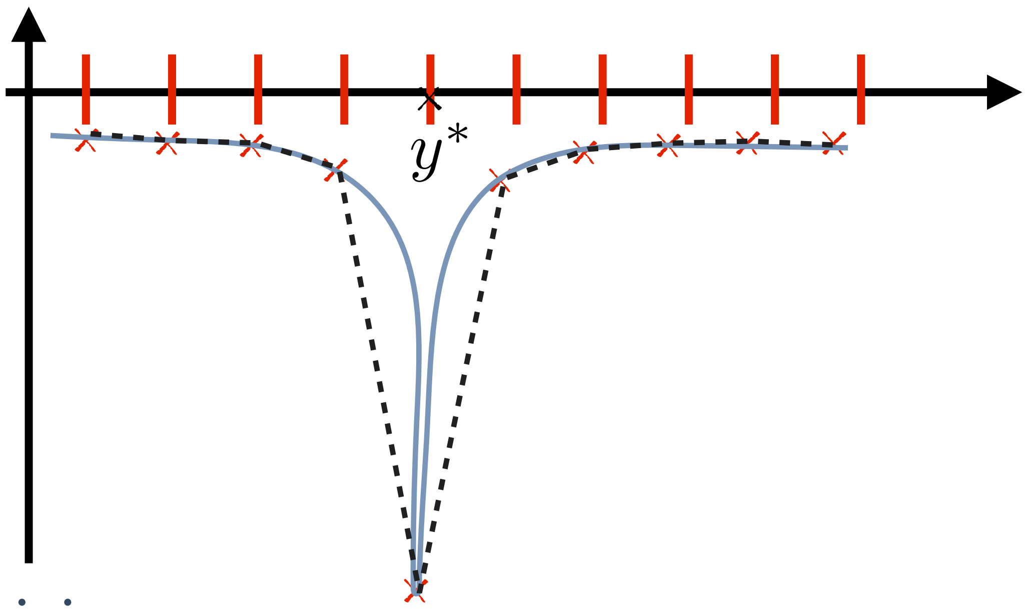

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3) Lots of techniques to achieve machine precision precision: regularization, interpolation, QBX, QB2X, …

Schwab, Wendland (1999), Beale et al. (2001, 2016), Helsing et al. (2008, 2018, 2022), Barnett (2014), Barnett et al. (2015), Perez-Arancibia et al. (2018, 2019), af Klinterbeg, Tornberg (2016, 2017, 2020), Wala, Klöckner (2018, 2019), Ding et al. (2021), Weed et al. (2022), Arrieta et al. (2022), …

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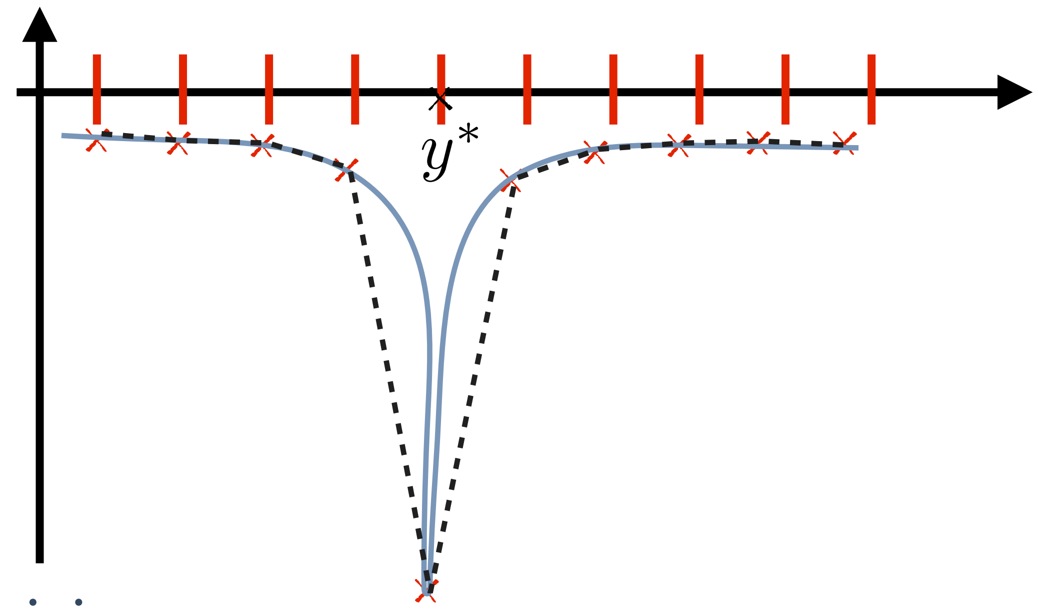

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Can we provide a simple method without 1), 2) (or 3)) ?

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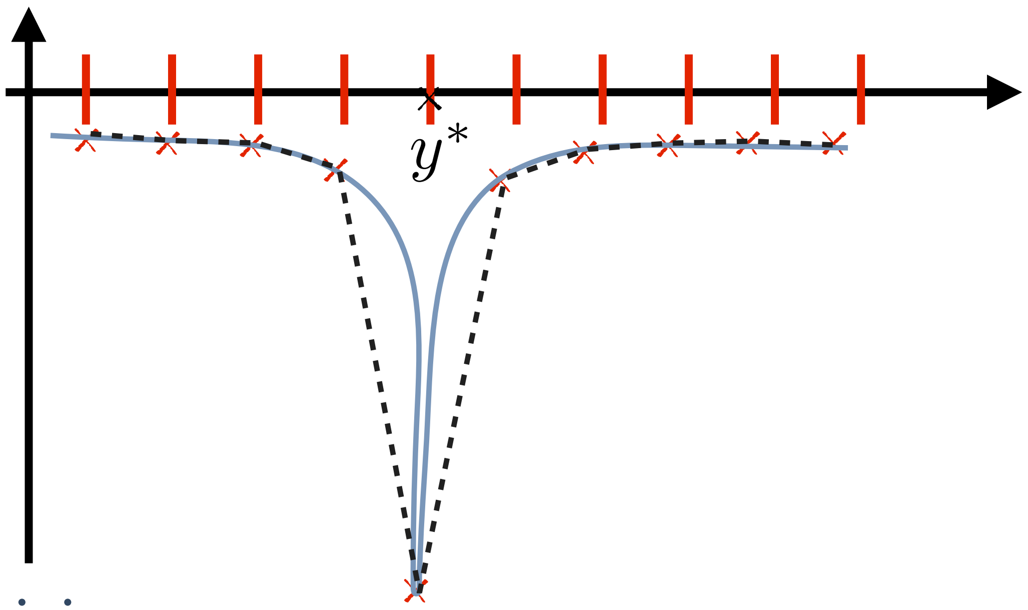
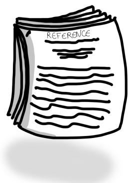


3) Lots of techniques to achieve machine precision precision: regularization, interpolation, QBX, QB2X, …



Schwab, Wendland (1999), Beale et al. (2001, 2016), Helsing et al. (2008, 2018, 2022), Barnett (2014), Barnett et al. (2015), Perez-Arancibia et al. (2018, 2019), af Klinterbeg, Tornberg (2016, 2017, 2020), Wala, Klöckner (2018, 2019), Ding et al. (2021), Weed et al. (2022), Arrieta et al. (2022), …

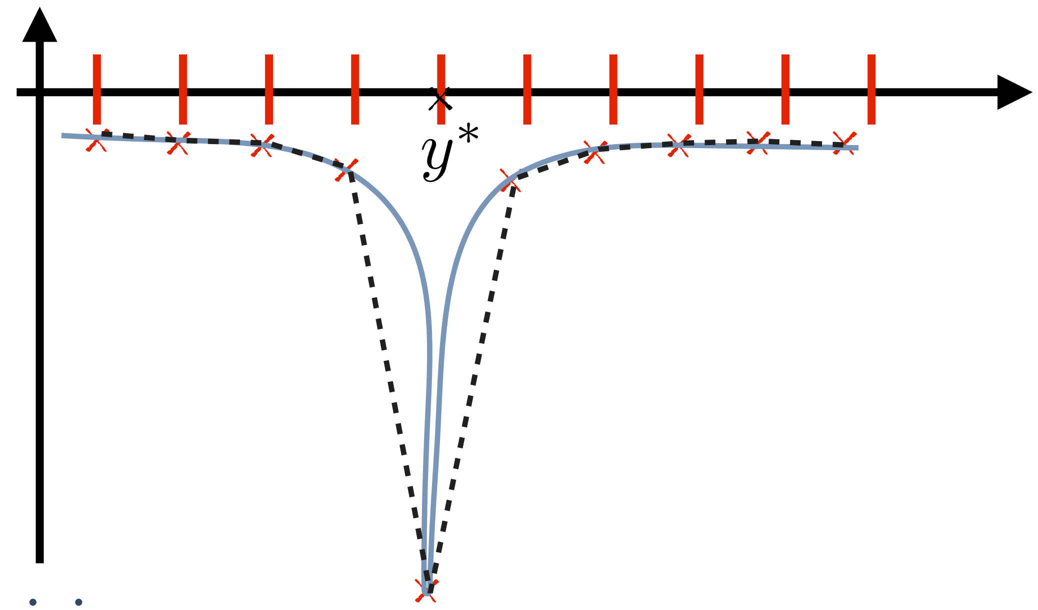
Can we provide a simple method without 1), 2) (or 3)) ?

Can we gain from it valuable insights into factors of the problem ?

How to address this error ?

1) increase the number of quadrature points
   ✘ close evaluation remains

2) use high-order Nyström methods
   ✘ requires sophisticated accelerated schemes



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Can we provide a simple method without 1), 2) (or 3)) ?

Can we gain from it valuable insights into factors of the problem ?

Given a quadrature method, can we improve accuracy at close evaluation points ?

Asymptotic methods for close evaluation

$$u(x) = \int_{\partial D} K(x,y)\mu(y)\, d\sigma_y$$ peaked kernel continuous function (spectral accuracy)

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1) Use asymptotic methods

-
-

2) Modified representations

-
-

C. et al. (2018, 2020, 2021), Khatri et al. (2020), C. (2021)

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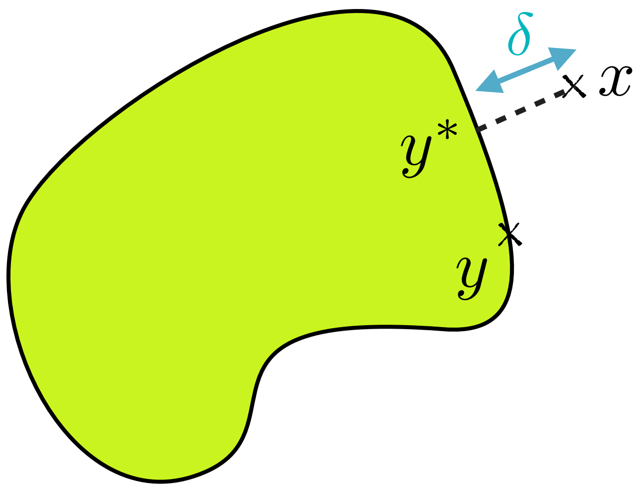

1) Use asymptotic methods

- $K = K^{\text{in}} + K^{\text{out}} + O(\delta)$
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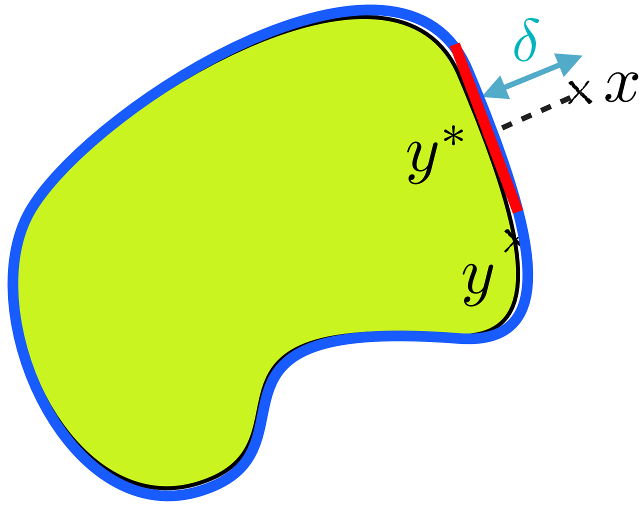

1) Use asymptotic methods

- $K = K^{\text{in}} + K^{\text{out}} + O(\delta)$
- $\int_{\text{in}} K(x,y)\mu(y)d\sigma_y + \int_{\text{out}} K(x,y)\mu(y)d\sigma_y$

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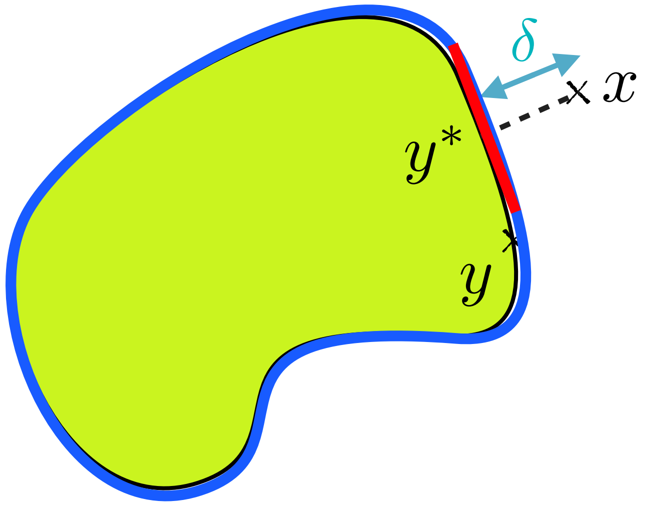

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Base numerical method on asymptotic analysis

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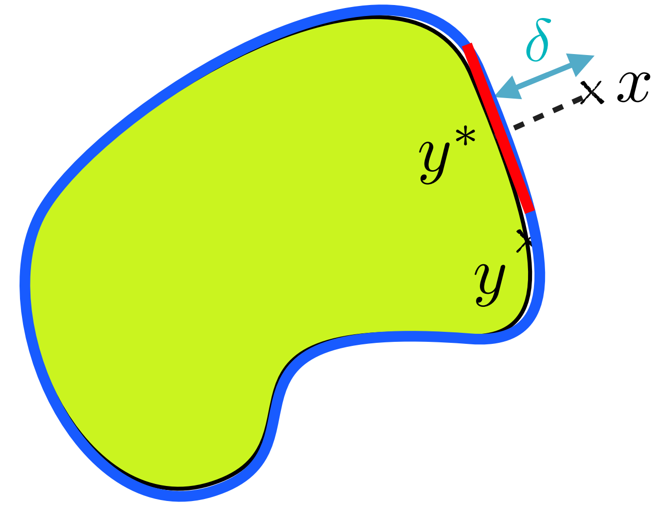

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INSA

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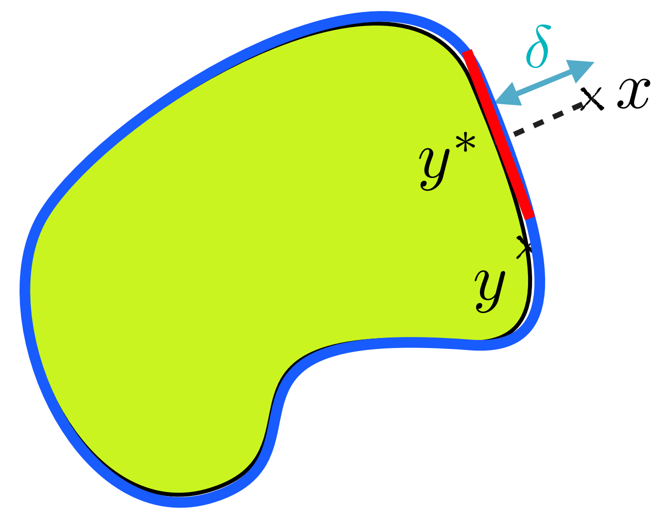

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Vanishes at $x = y$ Spectral computation

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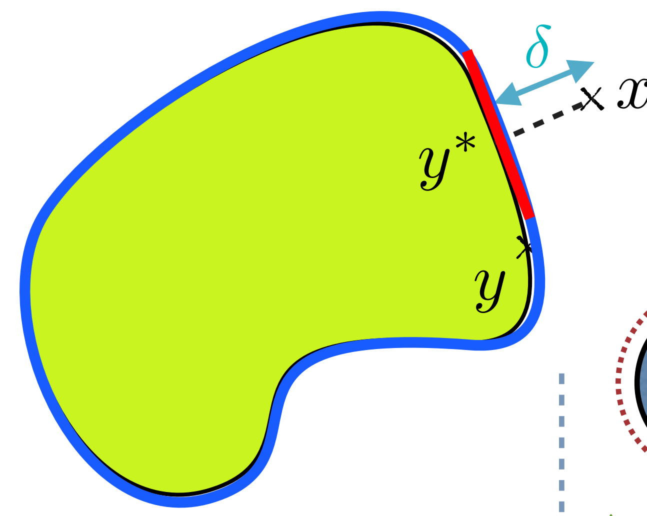

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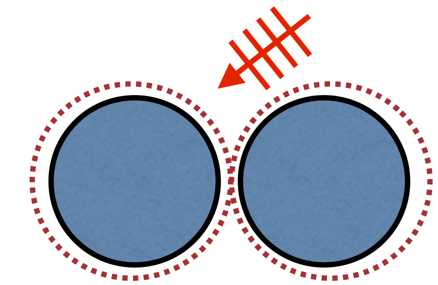
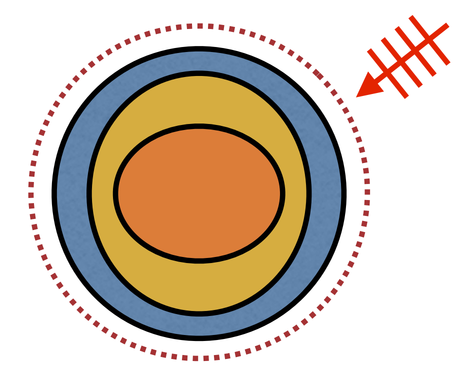
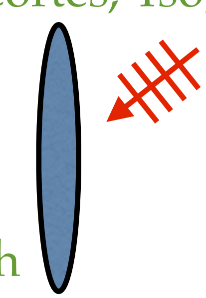


Acoustic binding with Kim, McCullough



Optical cloaking with Chaillat, Cortes, Tsogka



QPAX with Kim, Lewis, Moitier

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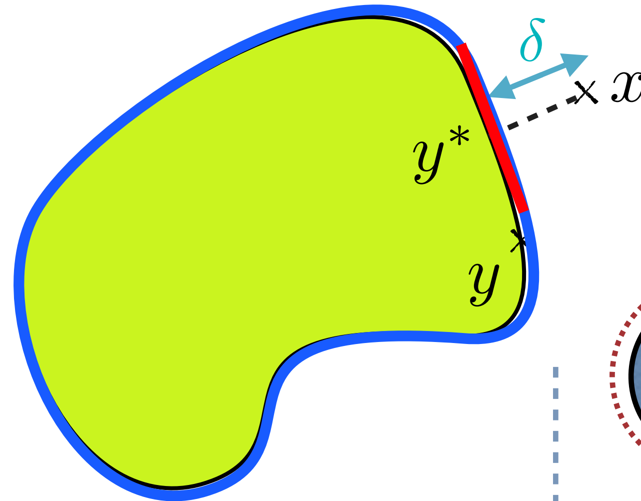

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Base numerical method on asymptotic analysis in 3D

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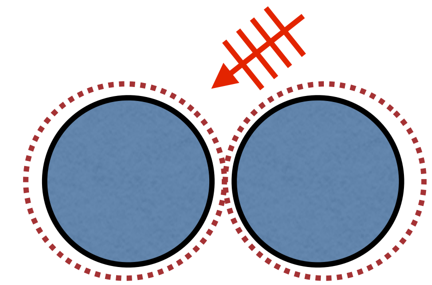
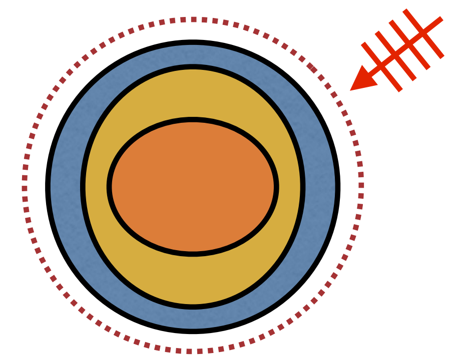


Acoustic binding with Kim, McCullough



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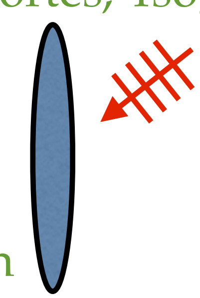


QPAX with Kim, Lewis, Moitier

Outline

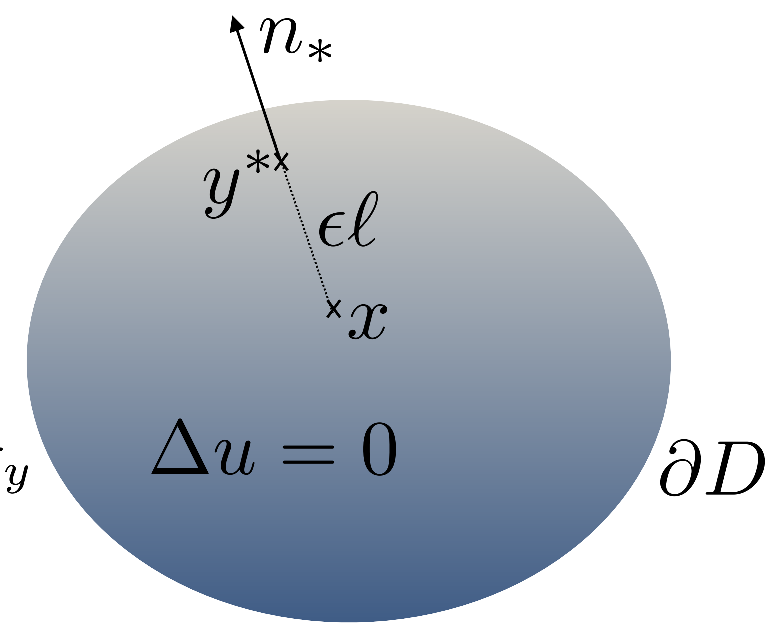
- The close evaluation problem
- Quadrature based on asymptotic methods
- Modified representations
- Conclusion

Nearly singular integrals

Interior Dirichlet Laplace problem

$$u(x) = -\int_{\partial D} \partial_{n_y} G(x,y) u(y) d\sigma_y + \int_{\partial D} G(x,y) \partial_n u(y) d\sigma_y$$

$$= -\frac{1}{4\pi}\int_{\partial D} \frac{n_y \cdot (x-y)}{|x-y|^3} u(y) d\sigma_y + \frac{1}{4\pi}\int_{\partial D} \frac{1}{|x-y|} \partial_n u(y) d\sigma_y$$



Nearly singular integrals

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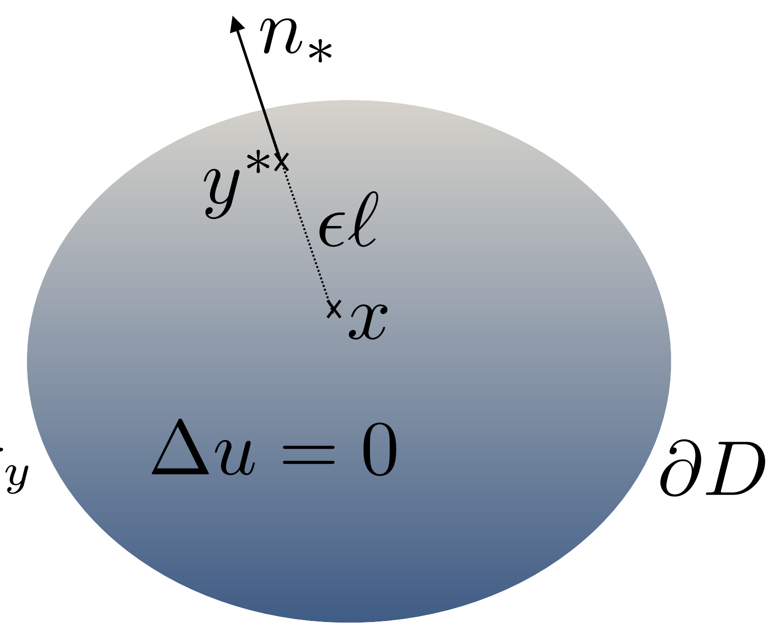

Using $x = y^* - \epsilon \ell n_*$ and Gauss's law:

$$\int_{\partial D} \partial_{n_y} G(x,y) d\sigma_y = \begin{cases} 0 & x \in \mathbb{R}^3 \setminus \bar{D} \\ -\frac{1}{2} & x \in \partial D \\ -1 & x \in D \end{cases}$$

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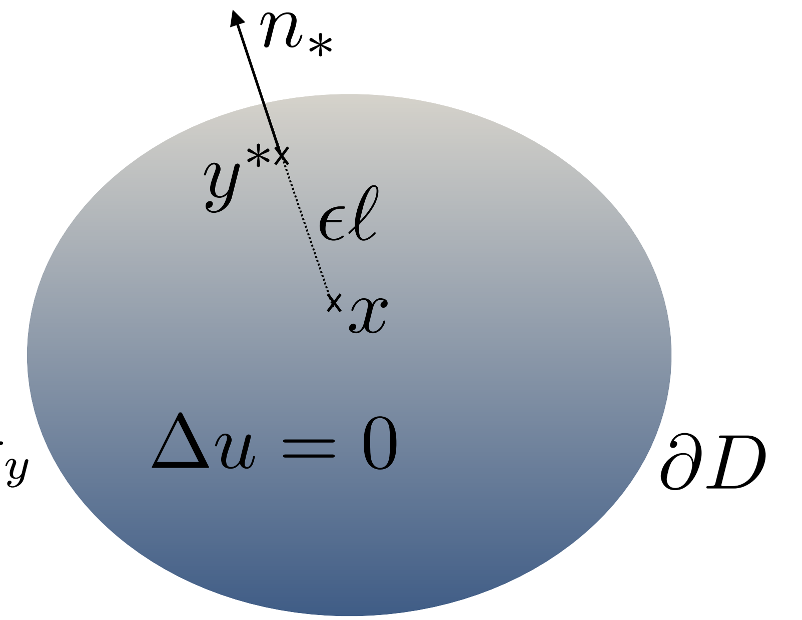

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Nearly singular integrals

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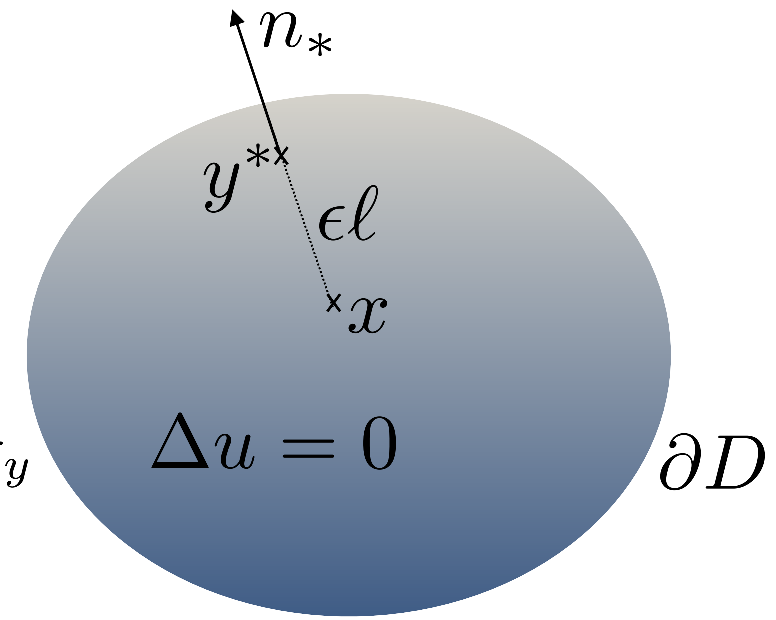

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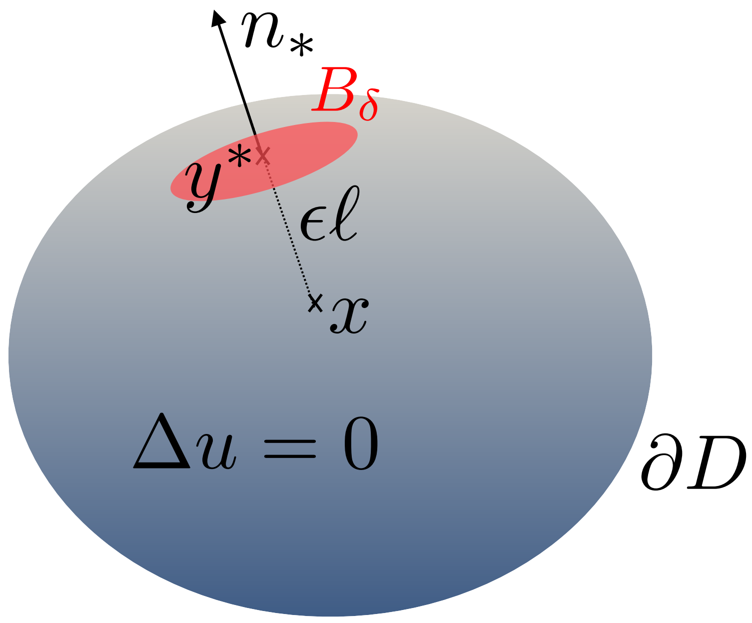
local analysis of each layer potential about y^* when $\varepsilon \to 0^+$

Close evaluation in 3D

Local Analysis of the Laplace double-layer potential:

$$-\frac{1}{4\pi}\int_{B_\delta}\frac{n_y\cdot(y^*-y-\epsilon\ell n_*)}{|y^*-y-\epsilon\ell n_*|^3}[u(y)-u(y^*)]d\sigma_y$$



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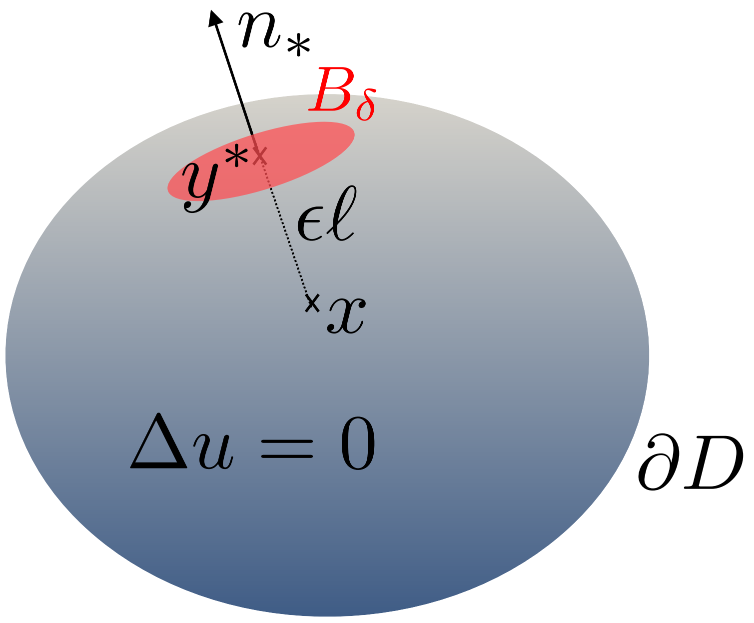
1) Parameterization $y(s,t)$ with $(s,t)\in[0,\pi]\times[-\pi,\pi]$ with y^* corresponding to the north pole



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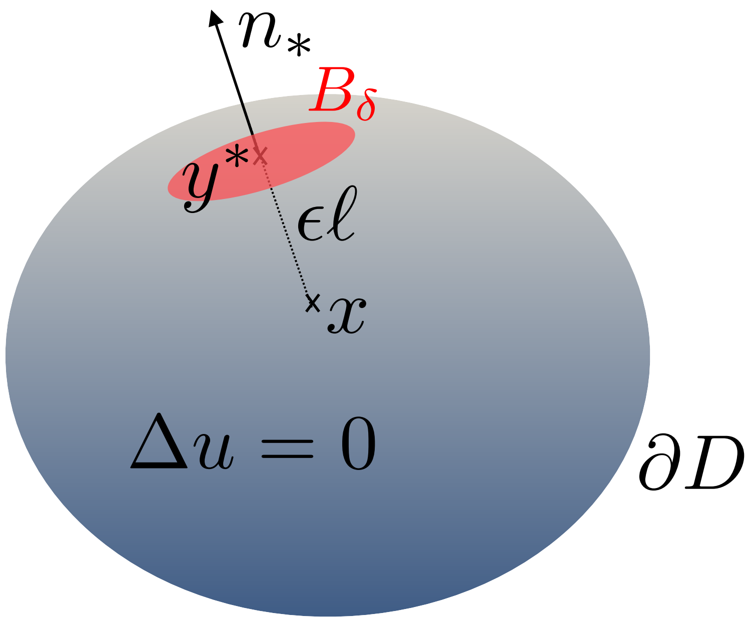

$$-\frac{1}{4\pi}\int_{B_\delta} \frac{n_y \cdot (y^* - y - \epsilon\ell n_*)}{|y^* - y - \epsilon\ell n_*|^3}[u(y) - u(y^*)]d\sigma_y$$

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$$J(s,t) = \frac{|y_s(s,t) \times y_t(s,t)|}{\sin(s)}$$

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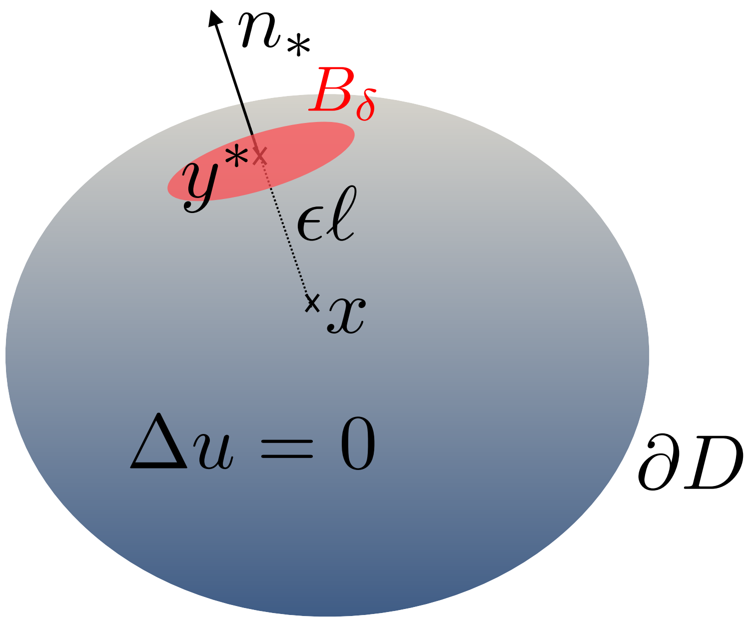

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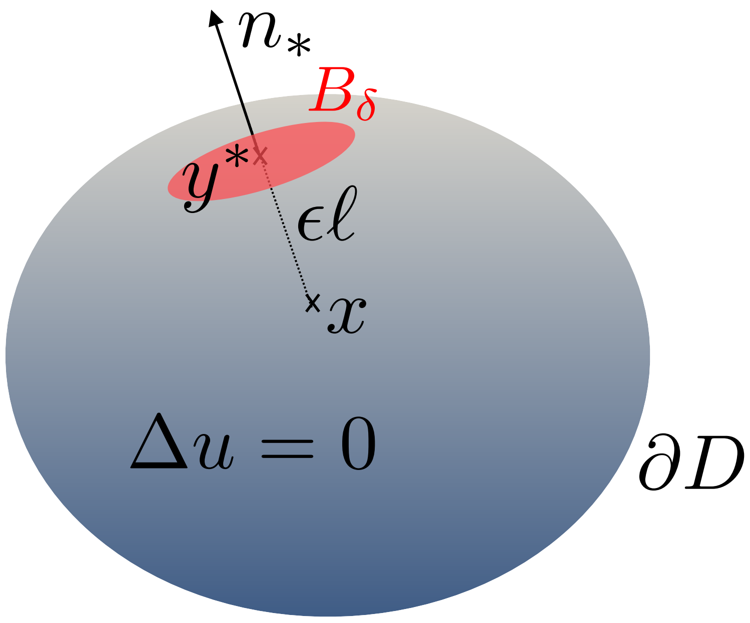
2) Let $s=\epsilon S$ and expand about $\epsilon=0$

Close evaluation in 3D



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2) Let $s = \epsilon S$ and expand about $\epsilon = 0$

$$\frac{1}{2}\int_0^{\delta/\epsilon}\left(\frac{\ell J(0,\cdot)S}{(S^2|y_s(0,\cdot)|^2 + \ell^2)^{3/2}} + O(\epsilon)\right)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{u}(\epsilon S,t) - \tilde{u}(0,\cdot)dt\right]dS$$

Close evaluation in 3D

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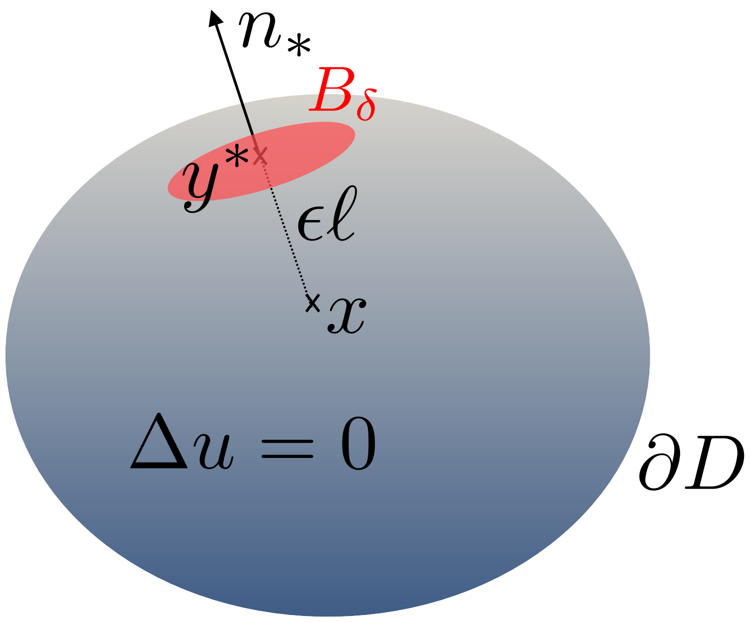

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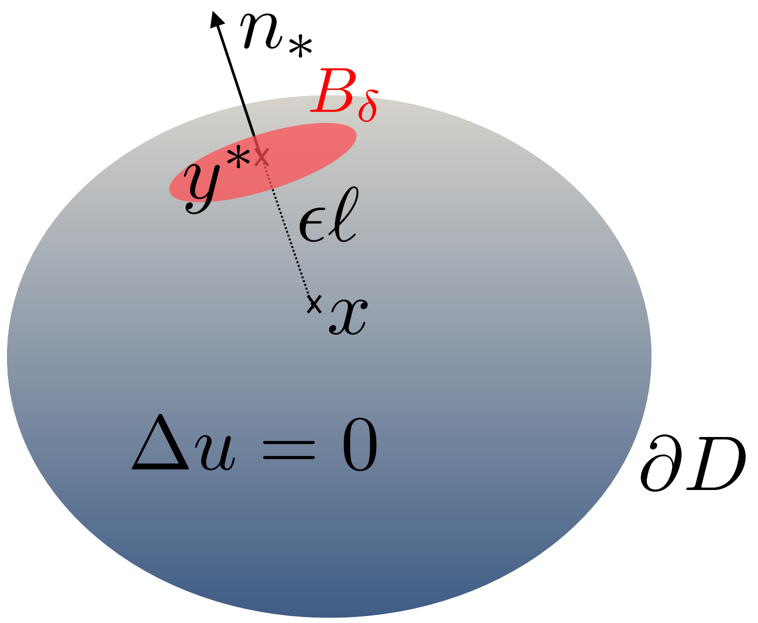
$$= \delta\frac{\epsilon\ell J(0,\cdot)}{8|y_s(0,\cdot)|^3}\Delta_S\tilde{u}(0,\cdot) + O(\epsilon^2)$$

Close evaluation in 3D

Local Analysis of the single-layer potential:

$$\frac{1}{4\pi}\int_{B_\delta}\frac{1}{|y^*-y-\epsilon\ell n_*|}\partial_n u(y)d\sigma_y$$



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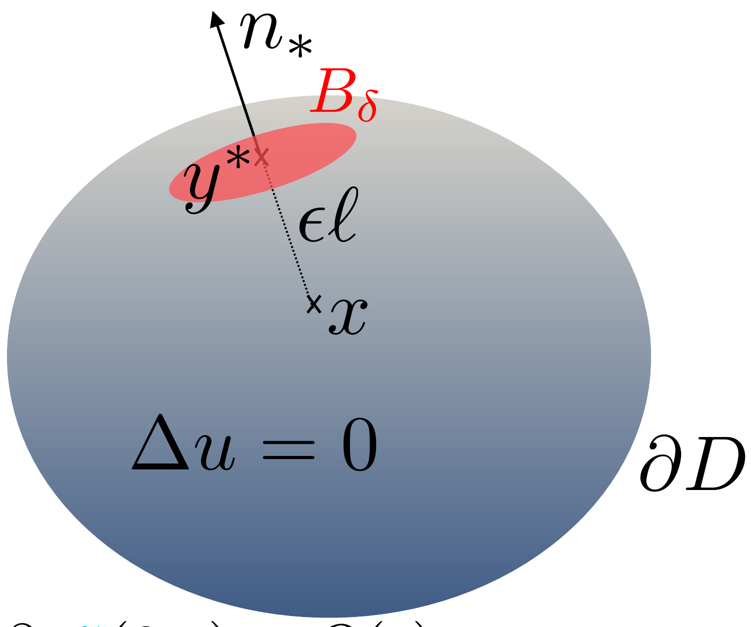

Steps 1) - 2)

$$\frac{1}{2}\int_0^{\delta/\epsilon} \left(\frac{J(0,\cdot)\epsilon S}{(S^2 |y_s(0,\cdot)|^2 + \ell^2)^{1/2}} + O(\epsilon^2) \right) \partial_n \tilde{u}(0,\cdot) dS = \delta \frac{J(0,\cdot)}{2|y_s(0,\cdot)|} \partial_n \tilde{u}(0,\cdot) + O(\epsilon)$$

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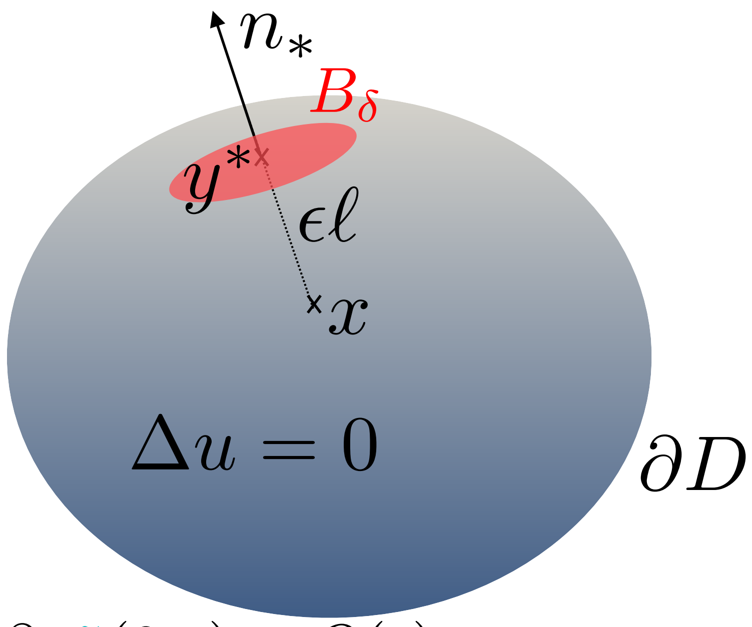

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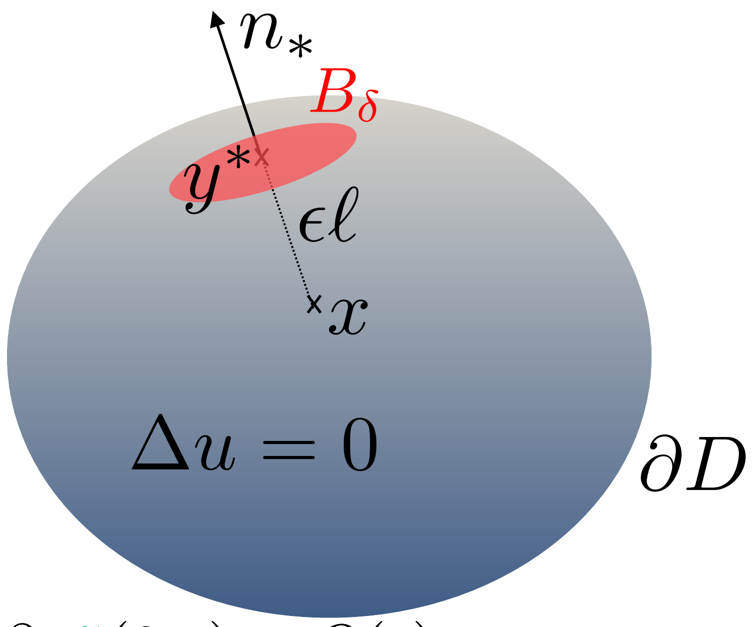

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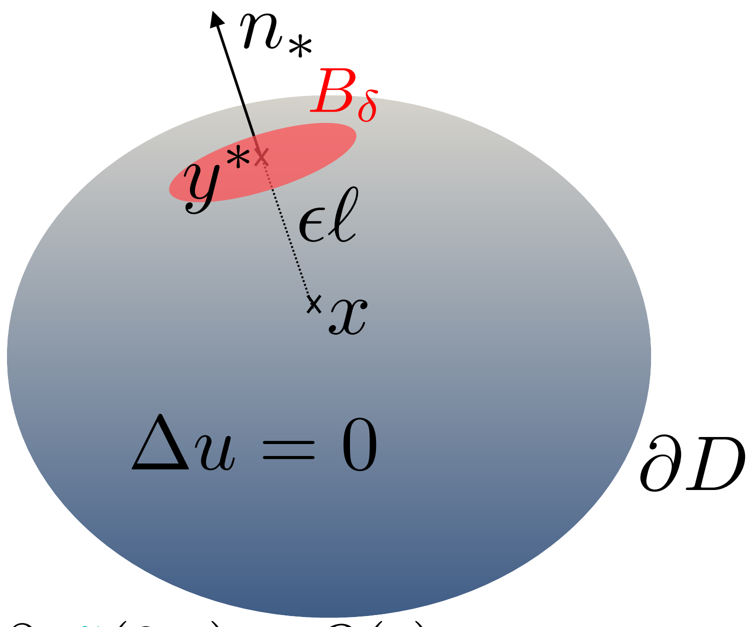

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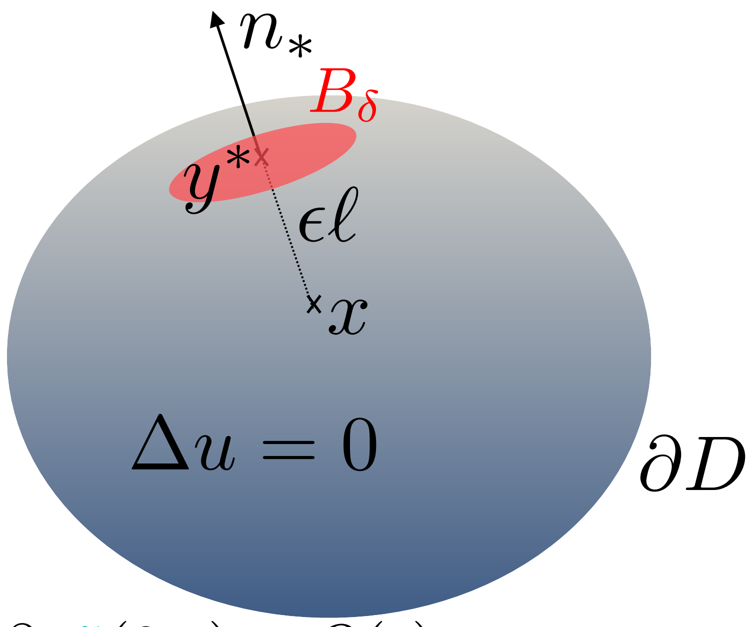

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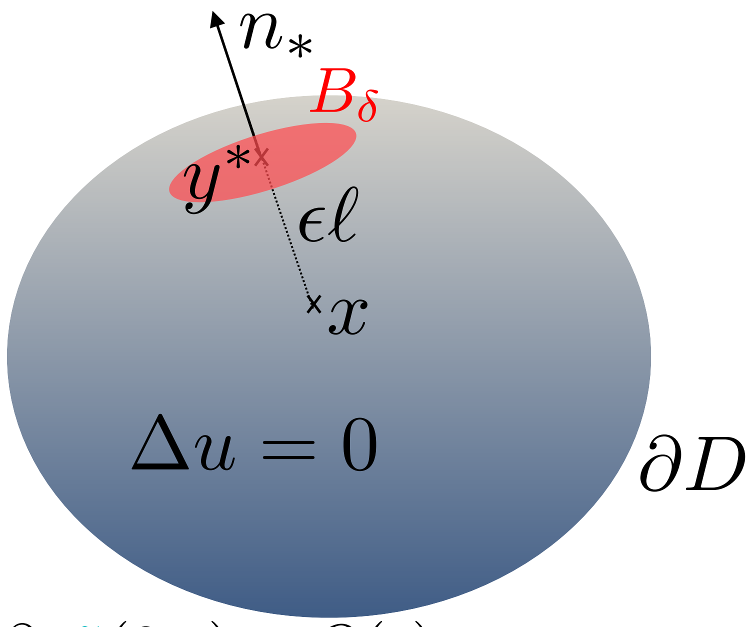

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Using the three above guidelines, how to proceed numerically ?

Numerical Method

Given a representation

$$u(y^*;\epsilon) = \int_{-\pi}^{\pi}\int_{0}^{\pi} F(s,t)\sin(s)\,dsdt$$

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Atkinson (1982), Gimbutas, Veerapaneni (2013)

2) N-point Gauss Legendre $z_i \in (-1,1),\ i = 1,\ldots,N$
with weights w_i

3) 2N Periodic Trapezoid rule
$t_j = -\pi + \pi\frac{j-1}{N},\ j = 1,\ldots,2N$

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$$u^N(y^*;\epsilon) = \frac{\pi}{8N}\sum_{i=1}^{N}\sum_{j=1}^{2N} w_i\sin(s_i)F(s_i,t_j)$$

Other Quadrature Rules

$$u^N(y^*;\epsilon) = \frac{\pi}{N}\sum_{i=1}^{N}\sum_{j=1}^{2N} w_i F(s_i, t_j)$$

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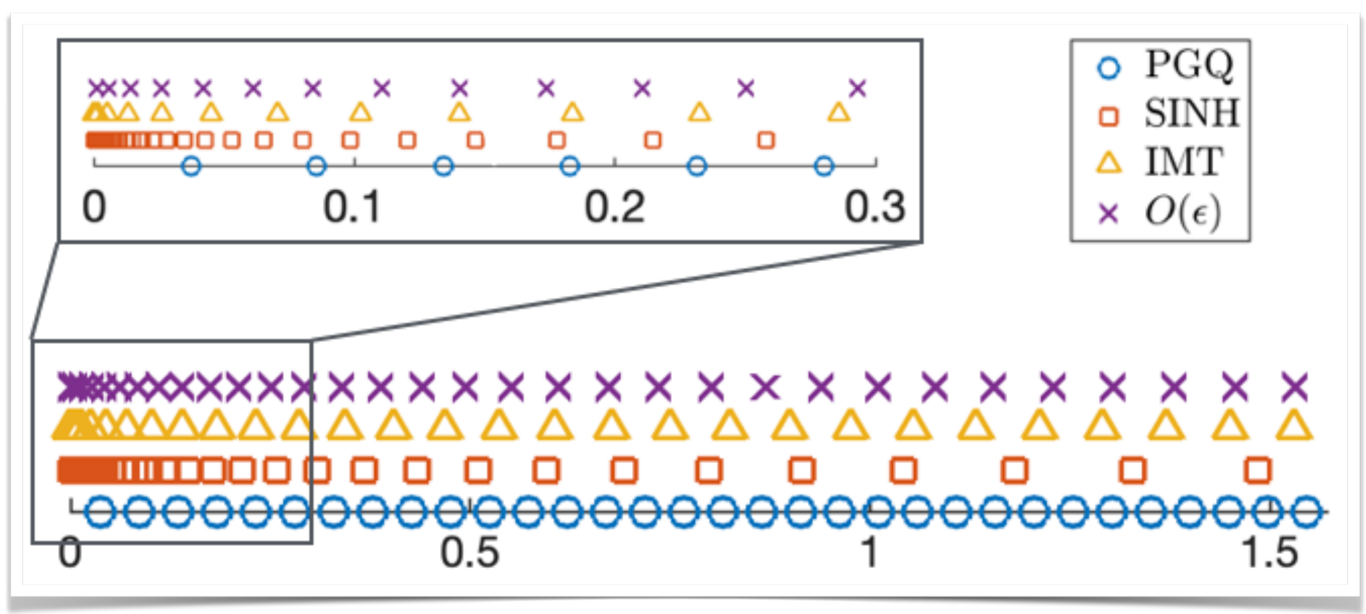


✔ The quadrature clusters points near y^*

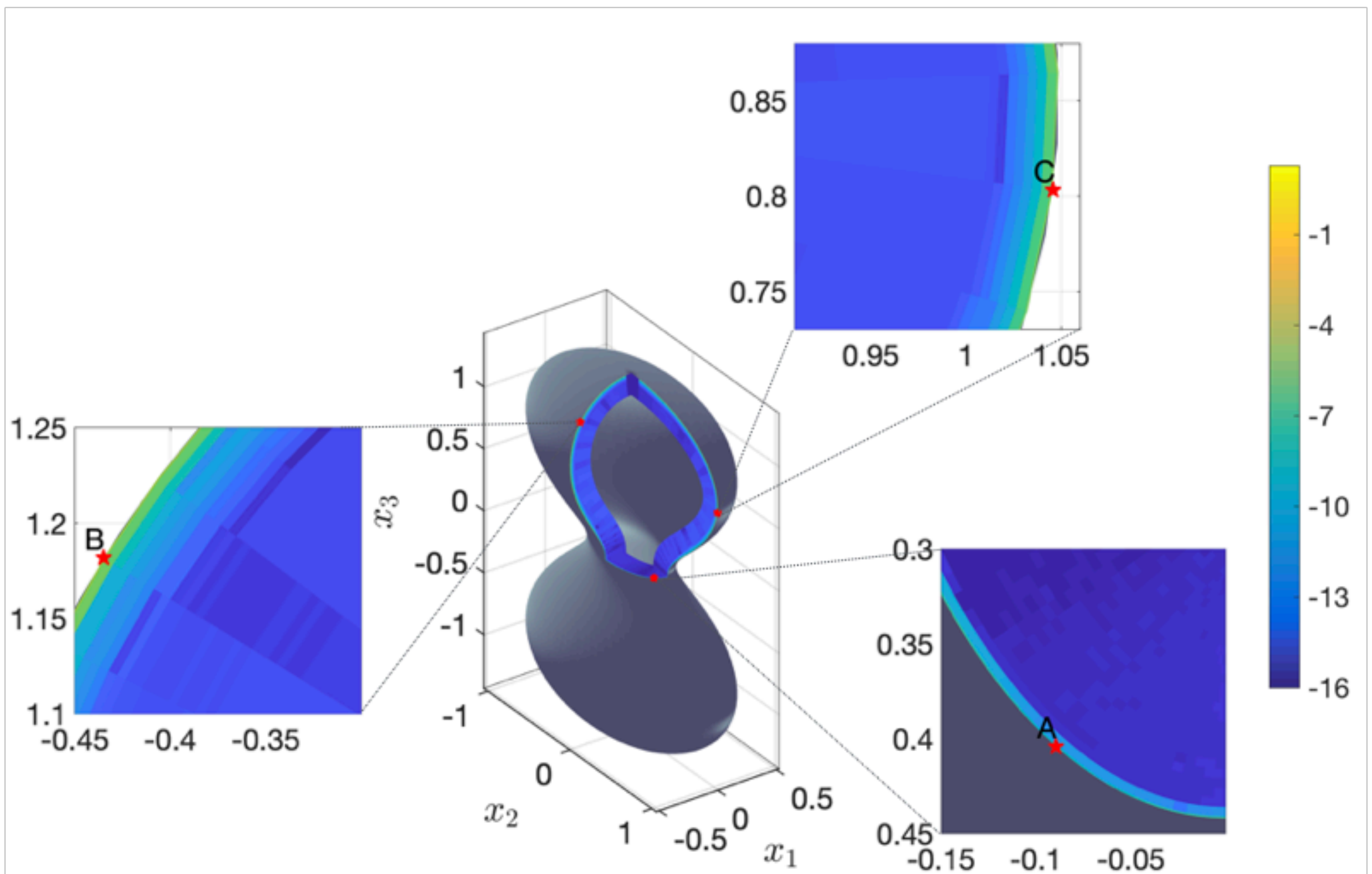
✔ The integrand is smoother at y^*

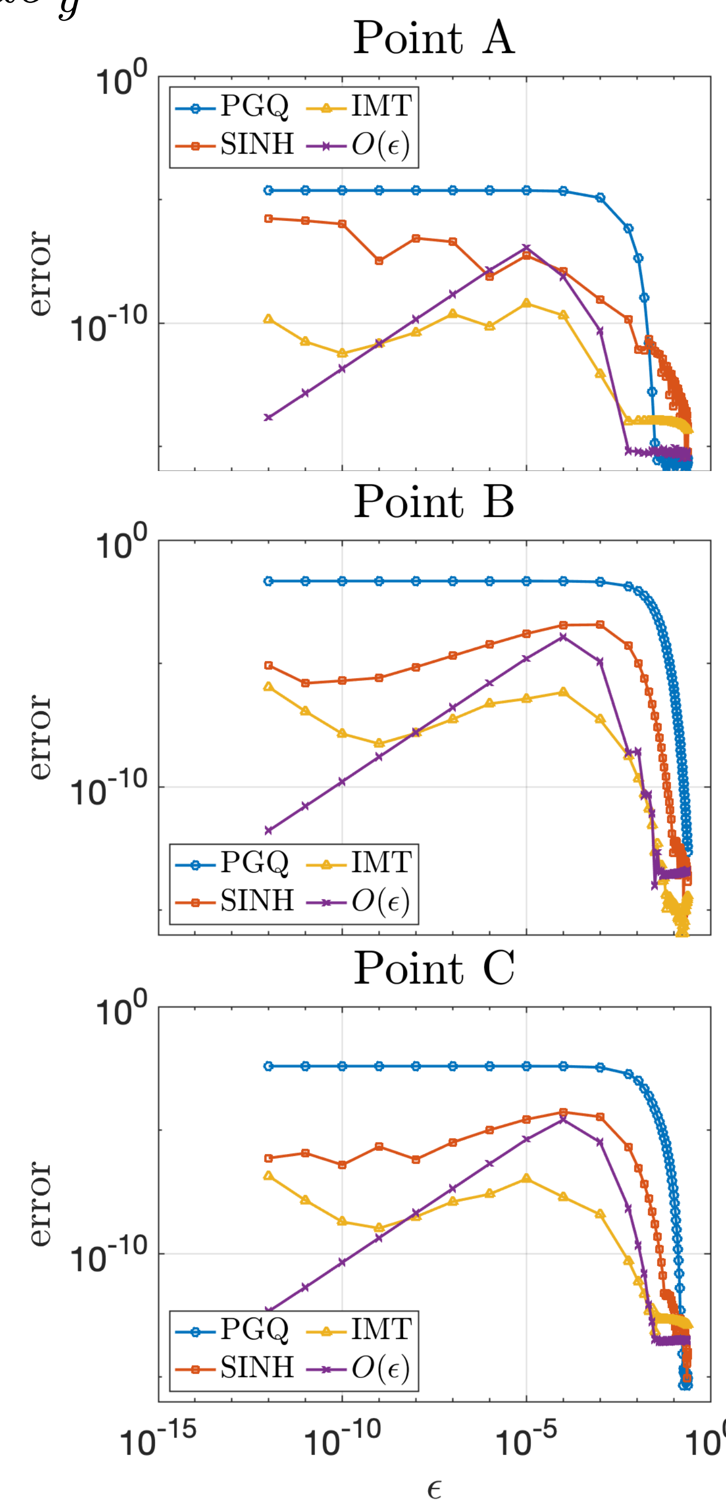
Numerical Results

$$u(x) = -\int_{\partial D} \partial_{n_y} G(x,y) u_{\text{ex}}(y) d\sigma_y + \int_{\partial D} G(x,y) \partial_n u_{\text{ex}}(y) d\sigma_y$$

Exact solution: $u_{\text{ex}}(x_1, x_2, x_3) = e^{x_3}(\sin x_1 + \sin x_2)$

Peanut shape with N = 128





Extension to $O(\epsilon^2)$

Peanut shape with N = 100

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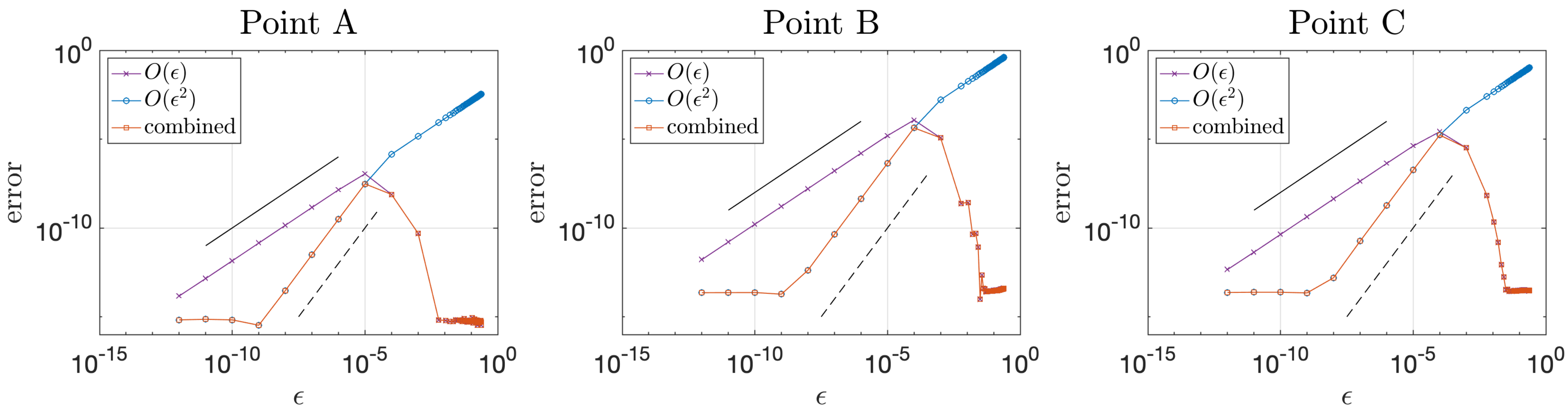
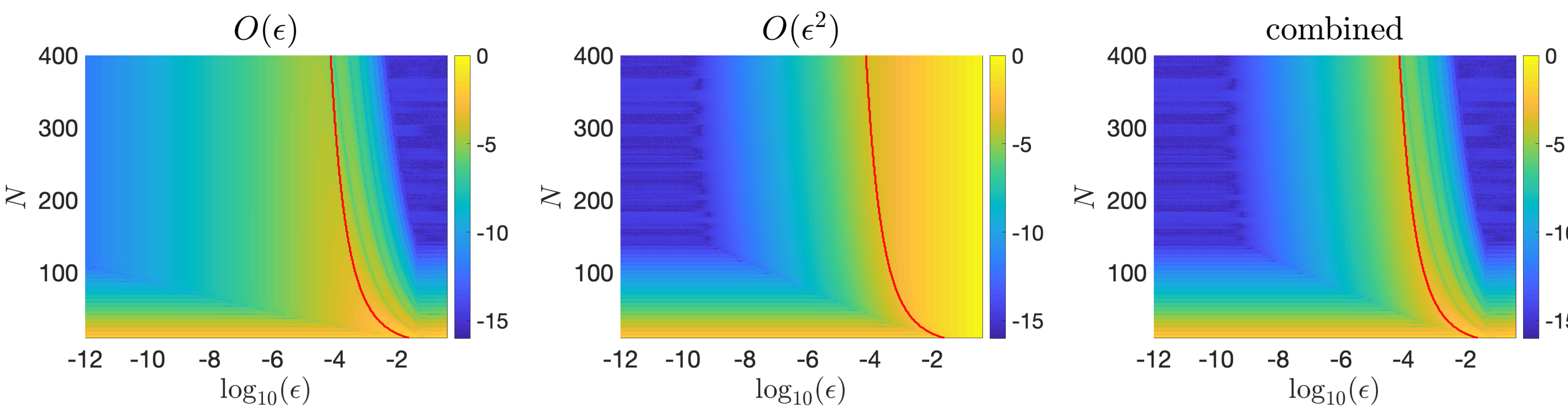
Use asymptotic approximation of the single-layer potential





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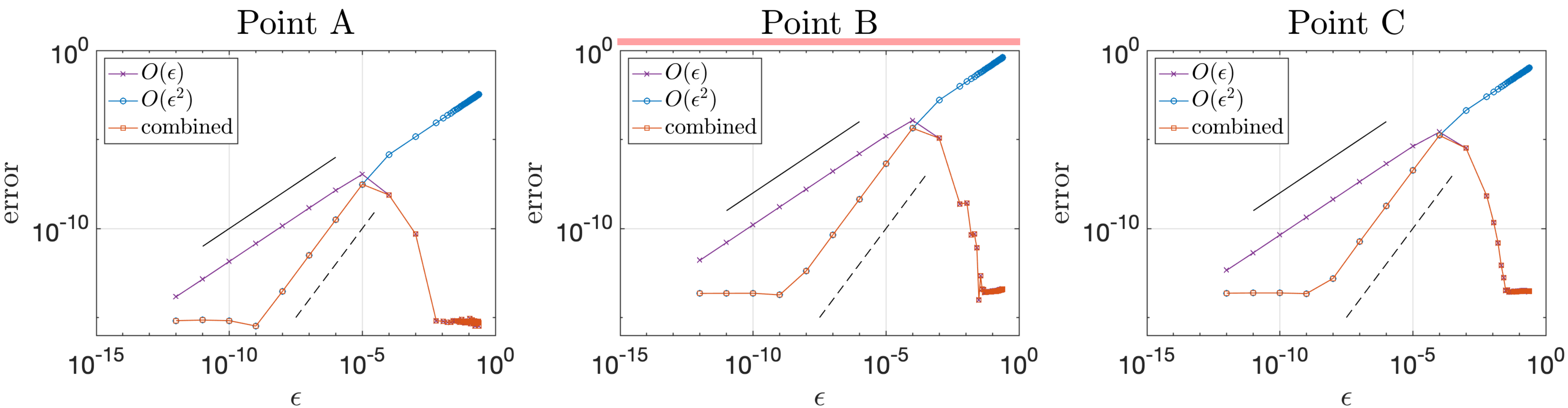
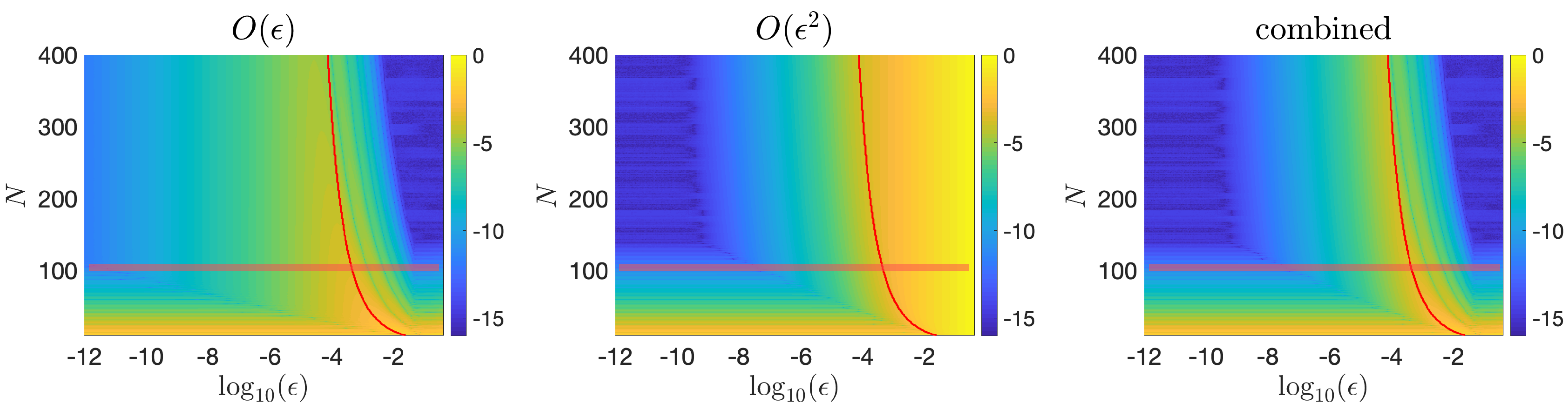
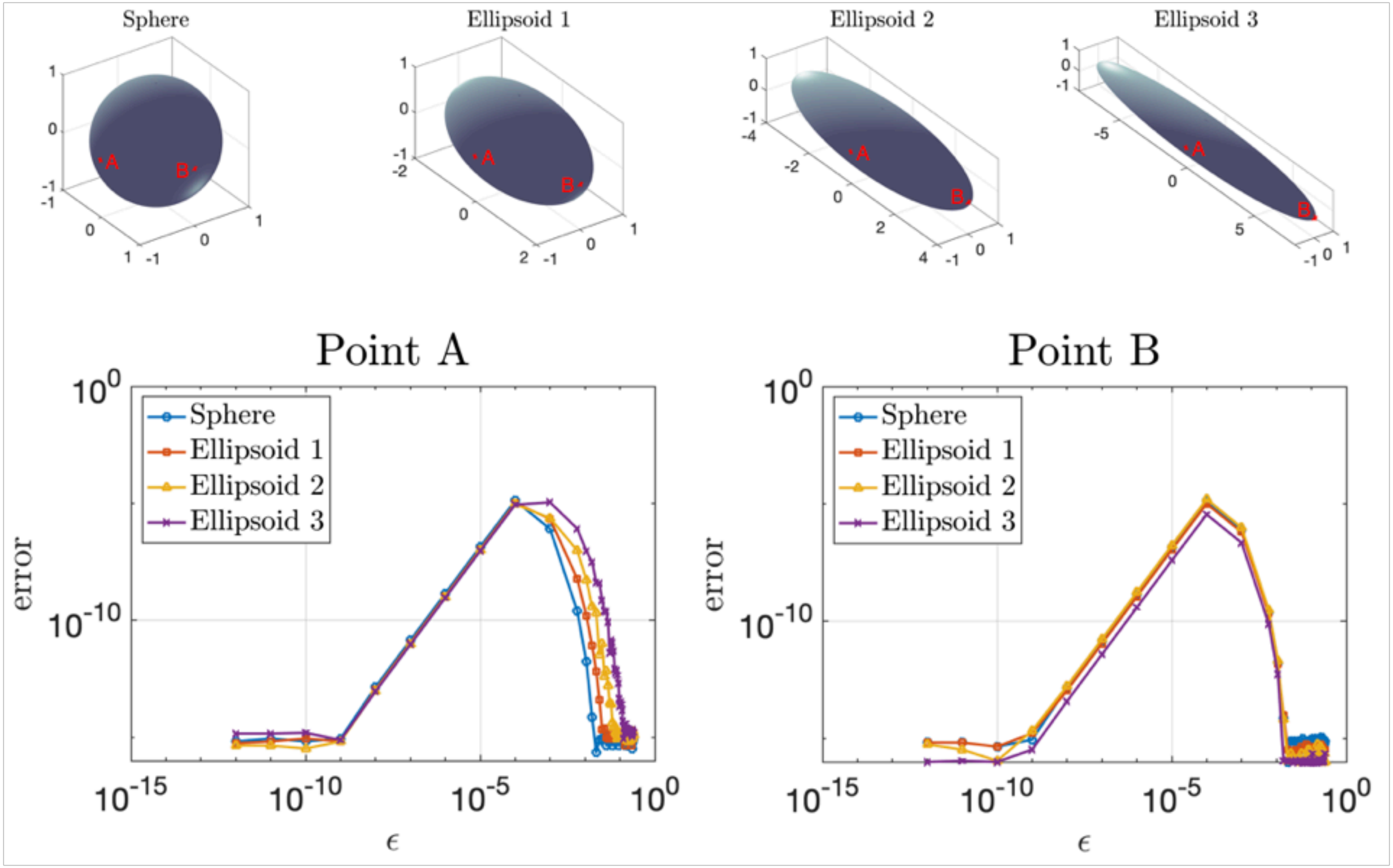




Effect of curvature



Outline

- The close evaluation problem
- Quadrature based on asymptotic methods
- Modified representations
- Conclusion

Modified representations

Previously we made use of Gauss' law to help attenuate the nearly singular behavior.

$$\int_{\partial D} \partial_n G(x,y)[u(y) - u(y^*)]d\sigma_y + u(y^*) \int_{\partial D} \partial_n G(x,y) d\sigma_y$$

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Representation formula gives us:

$$\int_{\partial D} \partial_{n_y} G(x,y)u(y)d\sigma_y - \int_{\partial D} G(x,y)\partial_n u(y)d\sigma_y = \begin{cases} -u(x) & x \in D \\ -\frac{1}{2}u(x) & x \in D \\ 0 & x \in \mathbb{R}^3 \setminus \bar{D} \end{cases}$$

G: fundamental solution (Laplace or Helmholtz)

u : solution (of Laplace or Helmholtz) in D

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Given $$\int_{\partial D} \partial_{n_y} G(x,y) u^{\mathrm{sol}}(y) d\sigma_y - \int_{\partial D} G(x,y) \partial_n u^{\mathrm{sol}}(y) d\sigma_y = \begin{cases} -u^{\mathrm{sol}}(x) & x \in D \\ -\frac{1}{2} u^{\mathrm{sol}}(x) & x \in D \\ 0 & x \in \mathbb{R}^3 \setminus \bar{D} \end{cases}$$

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Good resolution of ρ

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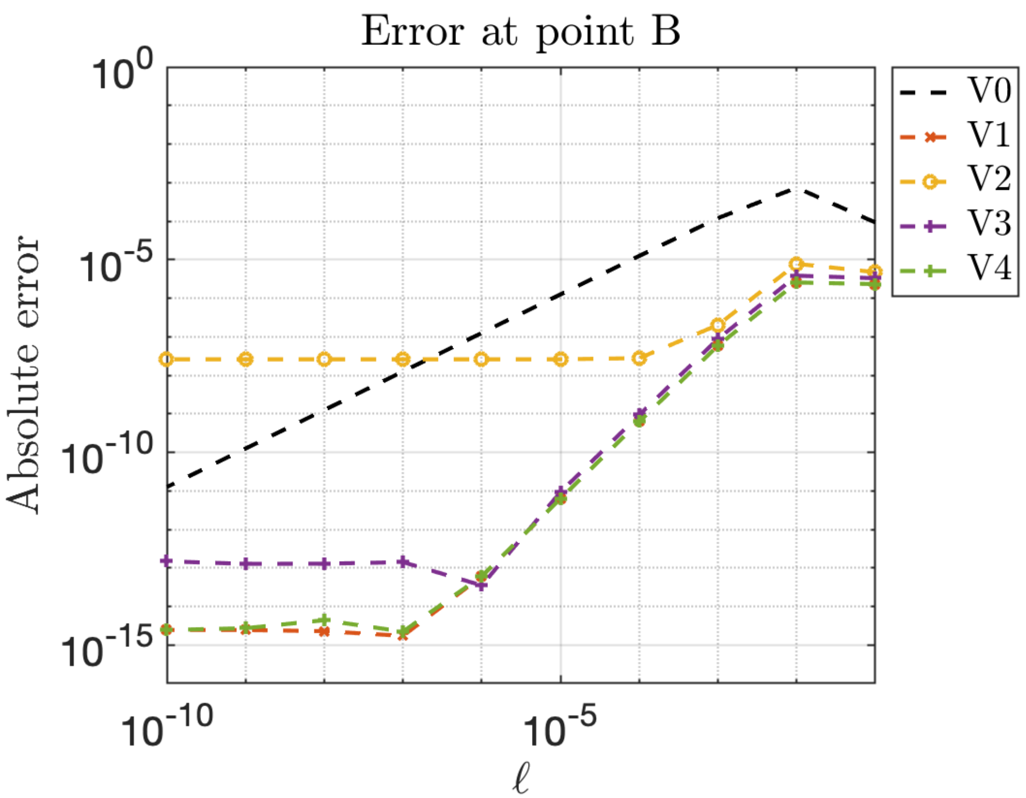
$\partial_n u^{\text{sol}}(y^*) = 1$

V1: $u^{\text{sol}}(y) = n^*\cdot y$

V2: $u^{\text{sol}}(y) = 4\pi G(y, y^*+n^*)$

V3: $u^{\text{sol}}(y) = \frac{1}{2}\frac{y_1^2-y_2^2}{n_{y^*,1}y_1^*-n_{y^*,2}y_2^*}$

V4: $u^{\text{sol}}(y) = \frac{(y_1-5)(y_2-5)}{n_{y^*,1}(y_2^*-5)+n_{y^*,2}(y_1^*-5)}$



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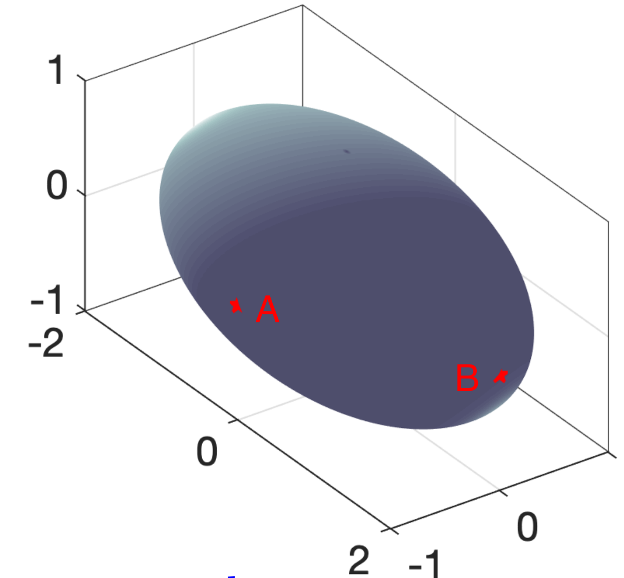
Results for Helmholtz layer potentials

Sound-soft scattering problem on an ellipsoid (k = 5)

Limited resolution of μ (10^{-7})



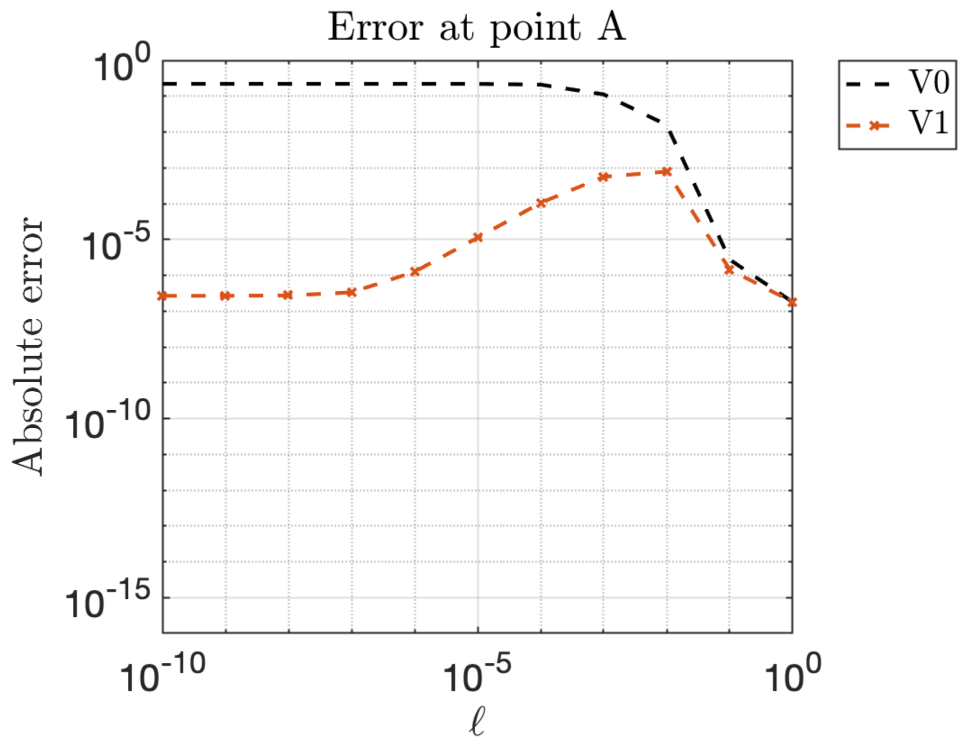
Representation V0: $u(x) = \int_{\partial D} [\partial_{n_y} G(x,y) - ikG(x,y)]\mu(y) d\sigma_y$

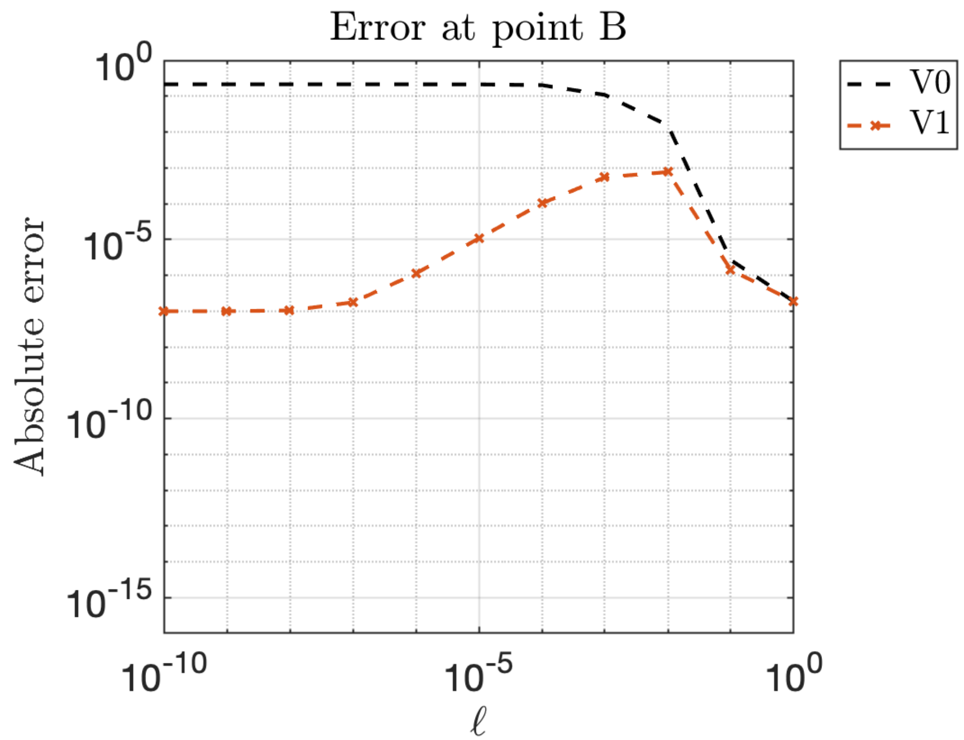
Representation V1: $u^{\mathrm{sol}}(y) = e^{ikn^* \cdot (y-y^*)}$

$$u(x) = \int_{\partial D} [\partial_{n_y} G(x,y) - \partial_n u^{\mathrm{sol}}(y) G(x,y)][\mu(y) - \mu(y^*)] d\sigma_y$$

$$+ \int_{\partial D} G(x,y)[\partial_n u^{\mathrm{sol}}(y) - ik]\mu(y) d\sigma_y + \mu(y^*) \int_{\partial D} \partial_{n_y} G(x,y)[1 - u^{\mathrm{sol}}(y)] d\sigma_y$$

N = 32





Outline

- The close evaluation problem
- Quadrature based on asymptotic methods
- Modified representations
- Conclusion

Summary

Due to sharply peaked behavior of layer potentials' kernel, one makes an O(1) error for close evaluation.

Local analysis provides valuable insights to design methods that naturally address the nearly singular behavior

When one has limited density resolution, modified representations help reduce the error (for free)

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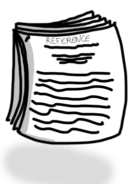
<u>Other techniques:</u>

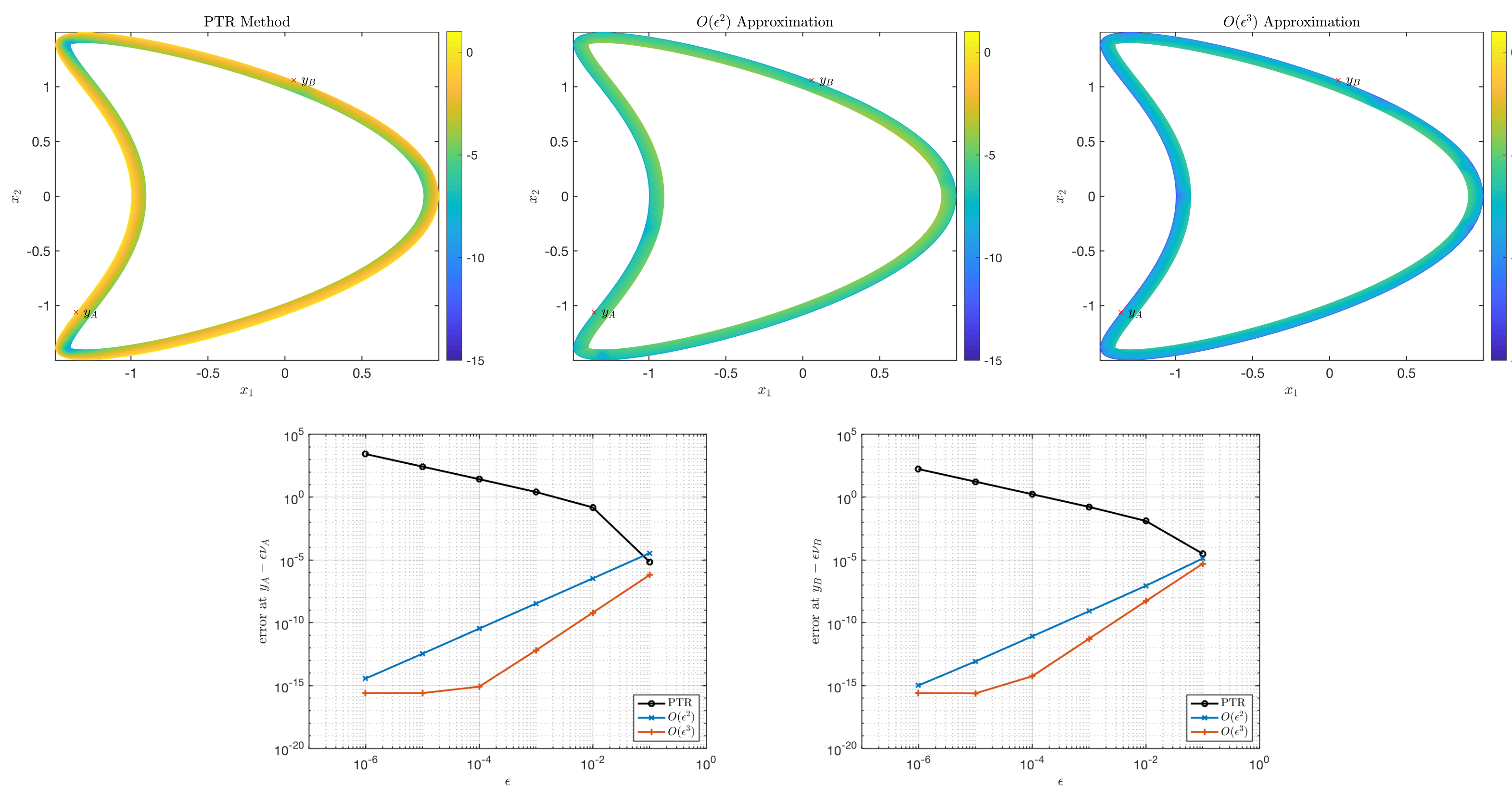
Kernel/singularity subtraction

Asymptotic approximations (2D)



Perez-Arancibia (2018)

Carvalho, Khatri, Kim (2020)



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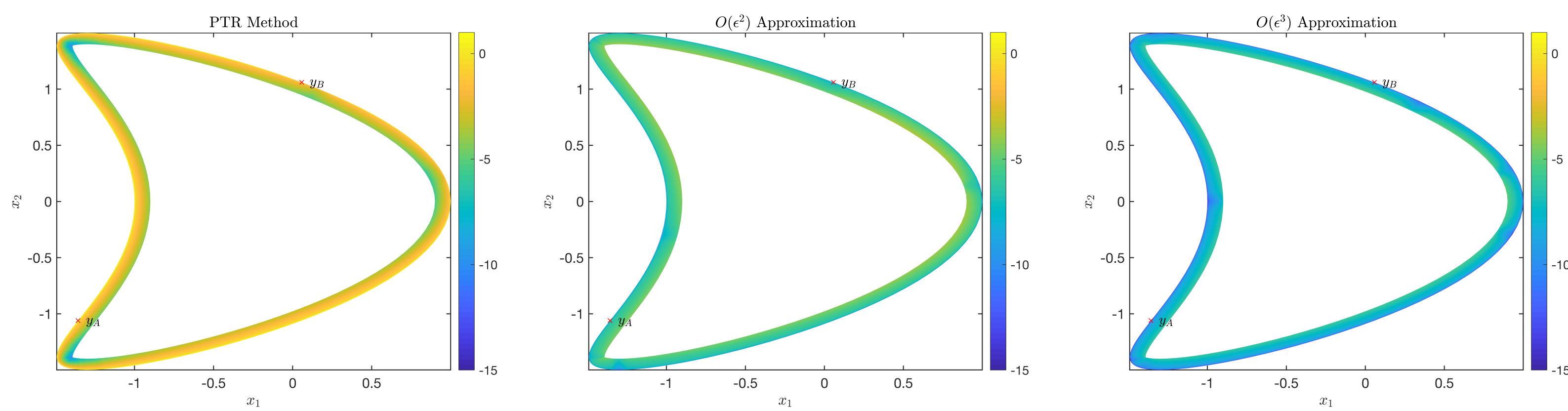

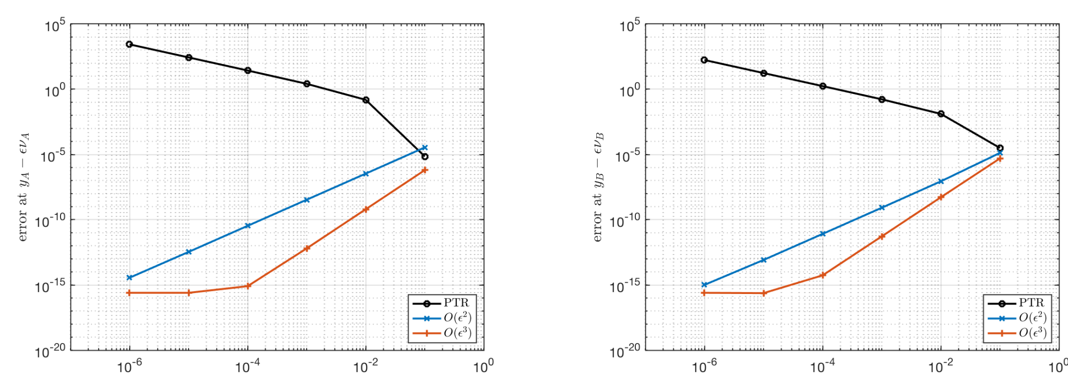

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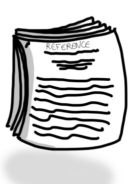

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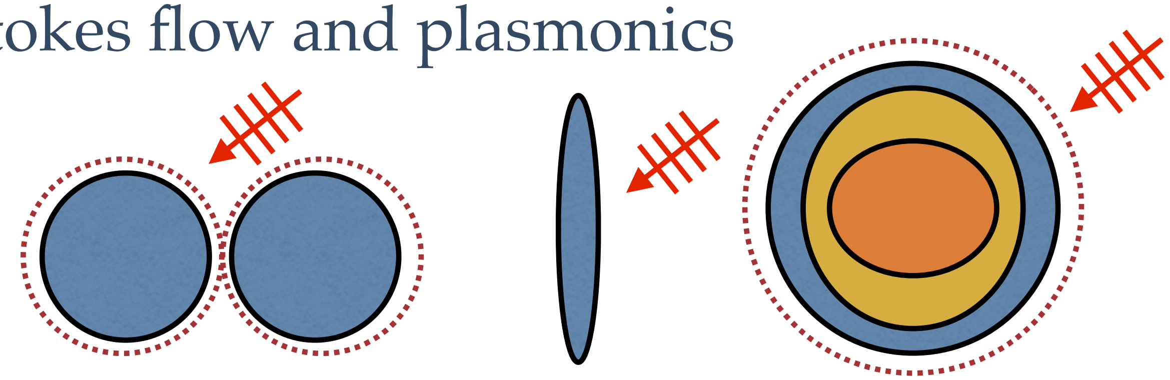
<u>Perspectives</u>:

Stokes flow and plasmonics

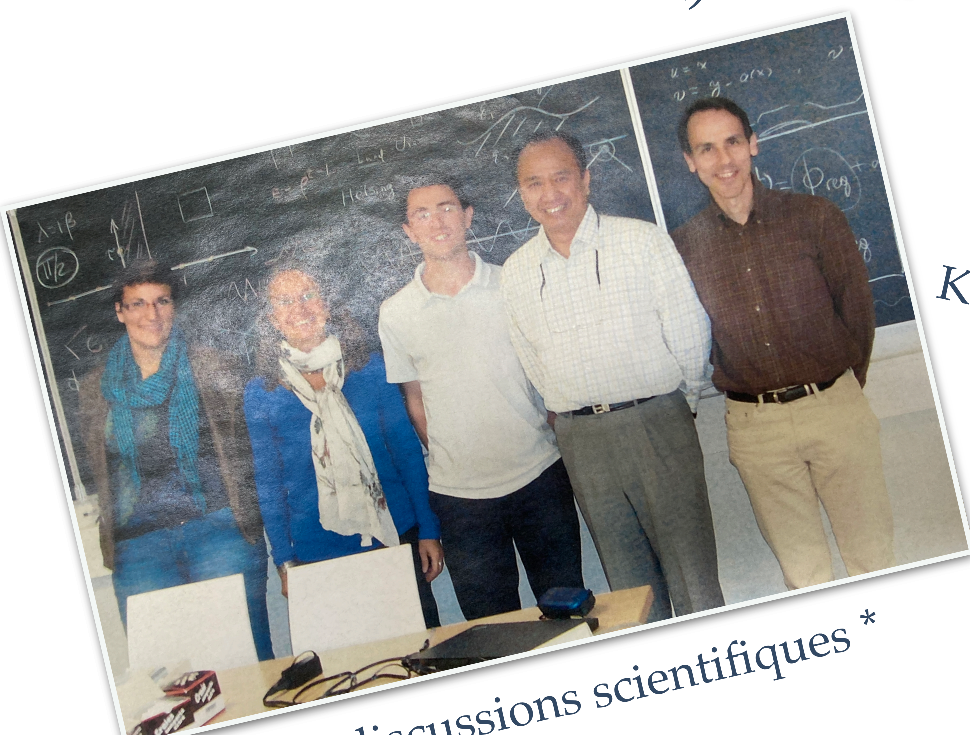


Thank you for your attention.

ETOPIM (Marseille 2012) *

CIRM (Marseille, 2022)

KOZ WAVES (New Castle/Sydney, 2014)



Des discussions scientifiques *

L'équipe de choc de l'UMA

* Photos de photos du bureau d'A.-S. Merci Lucas !