Une méthode de reconstruction pour un problème inverse en gravimétrie

Karim Ramdani

avec **Anthony Gerber-Roth & Alexandre Munnier**

UNIVERSITÉ DE LORRAINE

Inria

JO des POÈTES - RELAIS 4x60
17-19 avril 2024

Déambulation Ondulatoire

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17-19 avril 2024


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Asian Journal of Medicine and Health

18(9): 14-21, 2020; Article no.AJMAH.60013
ISSN: 2456-8414

Susceptibility: could age be the explanatory variable?

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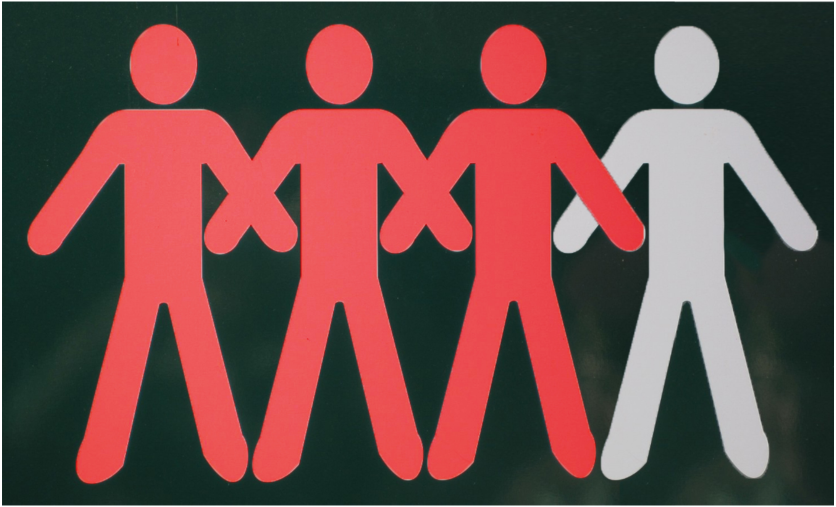
Authors' contributions

This work was carried out in collaboration among all authors. Author WO launched idea on Twitter, added some sentences, submitted the paper, corresponded with the kind publisher. Author MR



Après 60 ans, la susceptibilité augmente de 60% dans 3/4 des cas !

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Outline

1. Problem formulation

2. Reconstruction method
   - Step 1 : Constructing quadrature formula
   - Step 2 : Constructing quadrature domains

3. Numerical results

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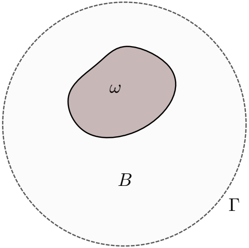
3. Numerical results

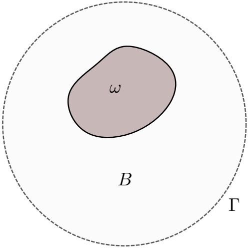
The inverse problem



The gravitational potential generated by a uniform mass distribution in ω satisfies

$$\begin{cases} -\Delta u_\omega = \mathbb{1}_\omega & \text{on } \mathbb{R}^2, \\ u_\omega = \mathcal{O}(\ln|x|) & |x| \to +\infty. \end{cases}$$

The inverse problem



Obviously, this gravitational potential is given by

$$u_{\omega}(x) = \int_{\omega} G(x-y)\ \mathrm{d}y = -(2\pi)^{-1} \int_{\omega} \ln|x-y|\ \mathrm{d}y$$

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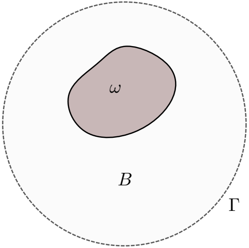

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Inverse problem

Reconstruct ω given u_ω (or ∇u_ω) on Γ.

Uniqueness

Theorem [Isakov, 1990]

Let ω_1, ω_2 be both

- star-shaped with respect to their centers of gravity

or

- convex in one direction

then $\nabla u_{\omega_1} = \nabla u_{\omega_2}$ on Γ implies $\omega_1 = \omega_2$.

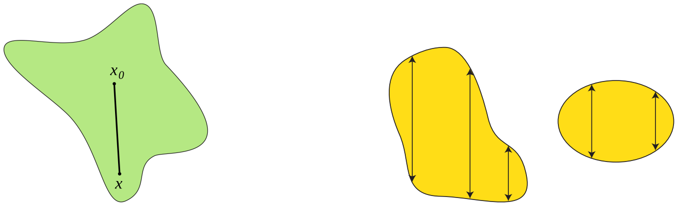

Figure – Star-shaped (left) and convex in one direction (right) domains.

Stability

Given $M, \rho_-, \rho_+, \alpha > 0$, let $\mathcal{U}$ be the set of star-shaped domains ω described in polar coordinates by

$$\omega = \left\{ x = (r, \theta) \in \mathbb{R}^2 \,|\, r < \rho(\theta), \quad \theta \in [0, 2\pi] \right\},$$

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Theorem, [Isakov, 1990]

There exists $C > 0$ such that for all $\omega_1, \omega_2 \in \mathcal{U}$,

$$\|\nabla u_{\omega_1} - \nabla u_{\omega_2}\|_{L^\infty(\Gamma)} < \varepsilon \Rightarrow \|\rho_1 - \rho_2\|_{L^\infty([0,2\pi])} \leqslant C|\ln \varepsilon|^{-\frac{1}{C}}.$$

Idea of the method

Our goal is to construct a sequence of domains $(\omega_N)_N$ satsifying the following (asymptotical) gravi-equivalence property :

$$\|\nabla u_\omega - \nabla u_{\omega_N}\|_{L^\infty(\Gamma)} \underset{N \to +\infty}{\longrightarrow} 0.$$

Idea of the method

By Green's formula, knowing u_ω and $\partial_n u_\omega := \nabla u_\omega \cdot n$ over Γ, we can write for all function v :

$$\int_\Gamma (u_\omega \, \partial_n v - \partial_n u_\omega \, v) = \int_{\Omega_0} \Delta v \, u_\omega - \int_{\Omega_0} \Delta u_\omega \, v.$$

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Using these harmonic moments as new measurements, we will construct the domains ω_N such that

$$\forall 0 \leqslant m \leqslant N, \quad \int_\omega z^m = \int_{\omega_N} z^m,$$

and these equalities will ensure the gravi-equivalence property.

Related problems

Calderón's inverse problem for highly conducting inclusions

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega_0 \setminus \overline{\omega}, \\ u = f & \text{on } \Gamma, \\ u = c & \text{on } \partial\omega, \end{cases}$$

where the constant c is such that : $\displaystyle\int_{\partial\omega} \frac{\partial u}{\partial n} = 0.$

Recovering the shape of a highly conducting inclusion ω from the knowledge of the DtN operator can also be formulated as a shape from moments inverse problem. See for instance the contributions of AMMARI *et al.* on **Generalized Polarisation Tensors** and the two papers by MUNNIER & R. (2017, 2018).

Related problems

Métrologie

Multiples	Symboles	Rapport à l'unité de mesure	Sous-multiples	Symboles	Rapport à l'unité de mesure
quetta-	Q	10^{30}	déci-	d	10^{-1}
ronna-	R	10^{27}	centi-	c	10^{-2}
yotta-	Y	10^{24}	milli-	m	10^{-3}
zetta-	Z	10^{21}	micro-	µ	10^{-6}
exa-	E	10^{18}	nano-	n	10^{-9}
péta-	P	10^{15}	**pico-**	p	10^{-12}
téra-	T	10^{12}	femto-	f	10^{-15}
giga-	G	10^{9}	atto-	a	10^{-18}
méga-	M	10^{6}	zepto-	z	10^{-21}
kilo-	k	10^{3}	yocto-	y	10^{-24}
hecto-	h	10^{2}	ronto-	r	10^{-27}
déca-	da	10^{1}	quecto-	q	10^{-30}

Outline

Reconstruction method

Step 1 Given $N \geqslant 1$, find the (complex) weights $c_1, ..., c_N$ and the nodes $z_1, ..., z_N$ such that :

$$\forall 0 \leqslant m \leqslant 2N-1, \quad \int_{\omega} z^m = \sum_{n=1}^{N} c_n z_n^m.$$

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Step 2 Knowing $c_1, ..., c_N$ and $z_1, ..., z_N$, construct (if possible) ω_N such that

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In particular, we will have

$$\forall 0 \leqslant m \leqslant 2N-1, \quad \int_{\omega} z^m = \int_{\omega_N} z^m.$$

Step 1 : Quadrature formula

Knowing

$$\tau_m = \int_\omega z^m,$$

we want to solve the nonlinear system of $2N$ equations and $2N$ unknowns (with pairwise distincts nodes)

$$\forall 0 \leqslant m \leqslant 2N-1, \quad \sum_{n=1}^{N} c_n z_n^m = \tau_m.$$

Such a system is known as a Prony's system, and appears for instance in the study of Padé's approximants, signal processing, error correction codes (see [Batenkov and Yomdin, 2013]).

Prony's method

Set

$$\mathbb{H}_0 = \begin{pmatrix} \tau_0 & \tau_1 & \cdots & \tau_{N-1} \\ \tau_1 & \tau_2 & \cdots & \tau_N \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{N-1} & \tau_N & \cdots & \tau_{2N-2} \end{pmatrix}$$

and

$$P(z) = \left| \begin{array}{cccc|c} & & & & \tau_N \\ & & & & \tau_{N+1} \\ & & \mathbb{H}_0 & & \vdots \\ & & & & \tau_{2N-1} \\ \hline 1 & z & \cdots & z^{N-1} & z^N \end{array} \right| .$$

Prony's method

Theorem

The Prony's system

$$\forall 0 \leqslant m \leqslant 2N-1, \quad \sum_{n=1}^{N} c_n z_n^m = \tau_m.$$

admits a solution if and only if the polynomial P admits N simple roots. In this case, this solution is unique and the nodes $z_1, \dots, z_N$ are the roots of P.

Prony's method

Knowing (z_n), we can compute (c_n) by solving the Vandermonde system

$$\mathbb{V}(z)\mathbf{c} = \boldsymbol{\tau}.$$

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Following [Golub, Milanfar and Varah, 1999], (z_n) are the eigenvalues of the generalized eigenvalue problem

$$\mathbb{H}_1\mathbf{x} = z\,\mathbb{H}_0\mathbf{x},$$

with

$$\mathbb{H}_0 = \begin{pmatrix} \tau_0 & \tau_1 & \cdots & \tau_{N-1} \\ \tau_1 & \tau_2 & \cdots & \tau_N \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{N-1} & \tau_N & \cdots & \tau_{2N-2} \end{pmatrix} \quad \mathbb{H}_1 = \begin{pmatrix} \tau_1 & \tau_2 & \cdots & \tau_N \\ \tau_2 & \tau_3 & \cdots & \tau_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_N & \tau_{N+1} & \cdots & \tau_{2N-1} \end{pmatrix}.$$

Prony's method

Remark

Solving Prony's system involves 2 ill-conditioned problems : a generalized eigenvalue problem with Hankel matrices and a Vandermonde linear system. To improve the conditioning, we solve a modified generalized eigenvalue problem

$$\boldsymbol{H}_1 \mathbf{x} = z\, \boldsymbol{H}_0 \mathbf{x},$$

where

$$\boldsymbol{H}_\ell = \Phi \mathbb{H}_\ell \Phi^T, \qquad \ell = 0, 1$$

and

$$\Phi = \text{Diag}\left[\frac{1}{\widehat{\rho}^n}\right]_{n=0}^{N-1}, \qquad \widehat{\rho} \sim |\tau_N|^{1/N}.$$

The unknown domain ω can be covered by plotting the disks centered at the nodes z_k with radii $\sqrt{|\mathrm{Re}(c_k)|/\pi}$.

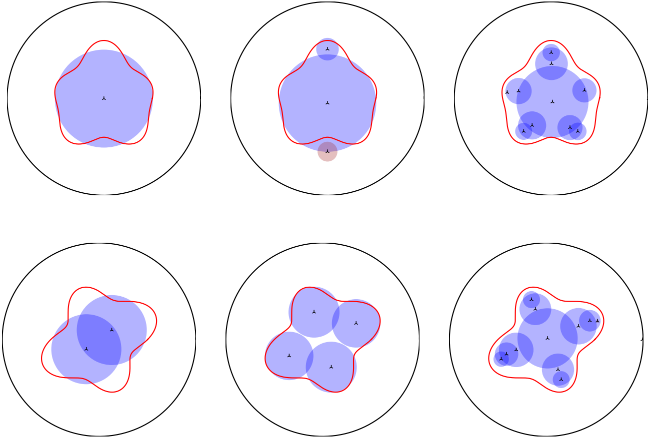

Figure – Covering the unknown domain by balls : two examples.

Step 2 : Quadrature domains

Given the weights $c_1, \dots, c_N$ and the nodes $z_1, \dots, z_N$, construct ω_N such that

$$\forall m \geqslant 0, \quad \int_{\omega_N} z^m = \sum_{n=1}^{N} c_n z_n^m.$$

This leads to the notion of **quadrature domains**.

Quadrature domains

Definition

Ω is a **Harmonic Quadrature Domain (HQD)** [1] *if there exists nodes $(z_n)_{1\leqslant n\leqslant N}$ in Ω and (real) weights $(c_n)_{1\leqslant n\leqslant N}$ such that for all harmonic function v :*

$$\int_\Omega v = \sum_{n=1}^{N} c_n v(z_n).$$

1. *"What is a quadrature domain ?"*, B. Gustafsson and H. S. Shapiro, 2005.

The **disk** is the simplest HQD, since for every harmonic function v in the disk $B(a,r)$ (mean value property)

$$\int_{B(a,r)} v = \pi r^2 v(a).$$

Quadrature domains

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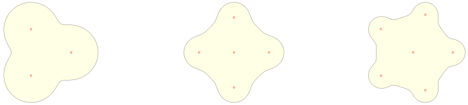

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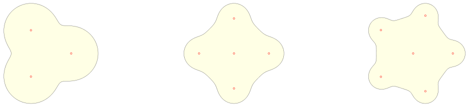

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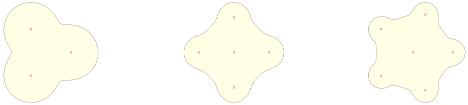

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- **Genreralization :** The above definition of HQD can be extended to arbitrary measures (and not only atomic).

Quadrature domains

Definition

Ω *is a* **Sub-Harmonic Quadrature Domain (SHQD)** *if there exists nodes* $(z_n)_{1\leqslant n\leqslant N}$ *in* Ω *and (positive) weights* $(c_n)_{1\leqslant n\leqslant N}$ *such that for every subharmonic function* v *(i.e.* $-\Delta v \leqslant 0$*) :*

$$\int_\Omega v \geqslant \sum_{n=1}^{N} c_n v(z_n).$$

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Ω is a **SHQD** $\Longrightarrow$ Ω is a **HQD** since

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In the class of **SHQD**, **existence and uniqueness** are ensured ([Gustafsson, 1990]).

Numerical construction of quadrature domains

The method proposed here is strongly related to partial balayage of measures [Gustafsson, Sakai, 1994] and obstacle problem [Gustafsson, Shahgholian, 1995].

We assume that the weights $c_1, \ldots, c_N$ are positive and that the nodes $z_1, \ldots, z_N$ are contained in the unit disk B.

Let Ω be the **SHQD** associated to these weights and nodes and assume that the Ω is compactly contained in B.

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The (unknown) function

$$u_\Omega(x) = -\frac{1}{2\pi} \int_\Omega \ln|x-y| \, \mathrm{d}y,$$

gives access to the characteristic function of Ω, since :

$$\mathbb{1}_\Omega = -\Delta u_\Omega.$$

Numerical construction of quadrature domains

It can be proved that u_Ω is the unique minimizer of the convex functional

$$\mathcal{E}(v) := \frac{1}{2}\int_B |\nabla v|^2 - \int_B v,$$

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Numerical construction of quadrature domains

To obtain the **SHQD** Ω, it suffices to compute u_Ω by solving the above (standard) minimization problem *, and use the fact that

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*. Computations are made with FreeFem++ and we use $\mathbb{P}_2$ finite elements.

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Remark

Since $G_N \notin H^1(B)$, we will use a slighlty modified (regularized) version of this result, that shows that $\mathbb{1}_\Omega = -\Delta \widetilde{u}_\Omega$ where $\widetilde{u}_\Omega$ is the unique minimizer of the convex functional $\mathcal{E}(v)$ over the closed convex set

$$\widetilde{K} := \{v \in H^1(B) \,|\, v \leqslant \widetilde{G}_N \text{ dans } B;\ v = \widetilde{G}_N \text{ sur } \partial B\},$$

with $\widetilde{G}_N(x) := -\dfrac{1}{2\pi} \displaystyle\sum_{n=1}^{N} \int_{B(z_n, r_n)} \ln|x-y| \, \mathrm{d}y$, $r_n := \sqrt{c_n/\pi}$.

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Algorithm (for positive weights)

1. Choose $N \geqslant 1$.
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$$\sum_{n=1}^{N} c_n z_n^m = \int_{\omega} z^m, \qquad \forall 0 \leqslant m \leqslant 2N-1.$$
4. **Step 2 :** Determine the unique **SHQD** ω_N associated to (z_n) and (c_n).

Convergence

Theorem

Assume that all the weights computed by the algorithm are positive.

1. If ω is a **SHQD** associated to a finite number of points, then there exists N such that $\omega_N = \omega$.
2. If there exists a compact set $K \subset B$ and a constant $C > 0$ such that $\omega_N \subset K$ for all $N \geqslant 1$:, then

$$\|\nabla u_{\omega_N} - \nabla u_{\omega}\|_{L^\infty(\Gamma)} \underset{N\to\infty}{\longrightarrow} 0.$$

If, in addition, ω_N and ω are star-shaped with respect to their centers of gravity and belong to $\mathcal{U}$, then ω_N converges to ω, in the sense that

$$\|\rho_{\omega_N} - \rho_{\omega}\|_{L^\infty(0,2\pi)} \longrightarrow 0.$$

Convergence

The equality determining the quadrature formula

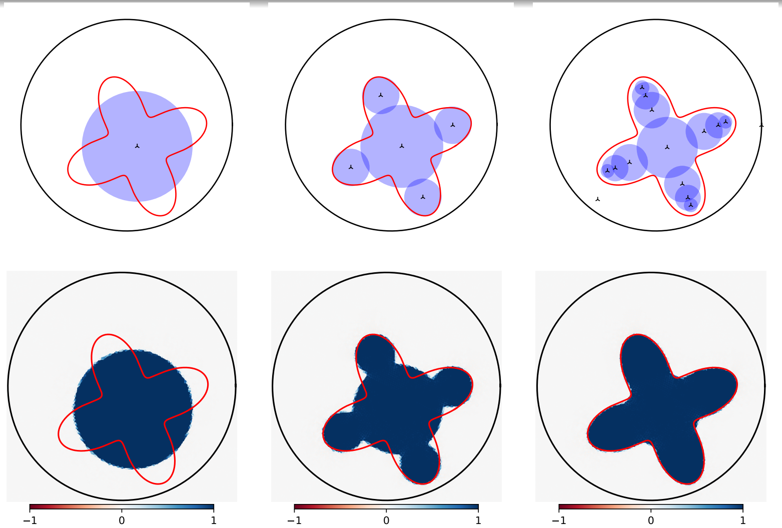
$$\int_{\omega} z^m = \sum_{n=1}^{N} c_n z_n^m, \qquad \forall 0 \leqslant m \leqslant 2N-1$$

can also be written

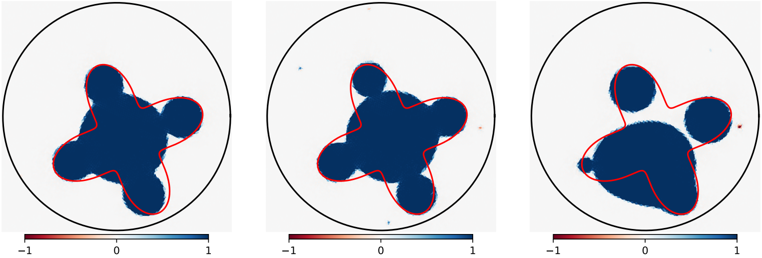
$$\int z^m \mathbb{1}_{\omega} = \int z^m \mathrm{d}\mu_N, \quad \mu_N := \sum_{n=1}^{N} c_n \delta_{z_n}.$$

It is thus natural to expect the weights to be positive. In most of our numerical examples, the weights were real and positive. When negative weights appear, one needs tu use another algorithm to construct **HQD** ; see [Ameur, Helmer and Tellander, 2021] and [Gerber-Roth, Munnier, Ramdani, 2023].

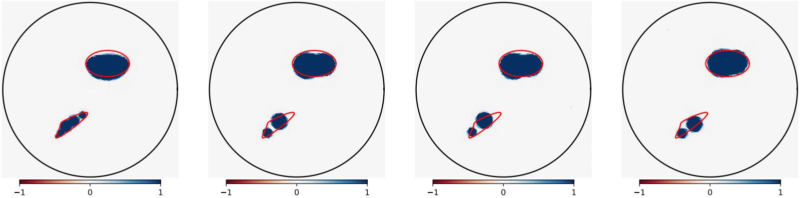
Outline



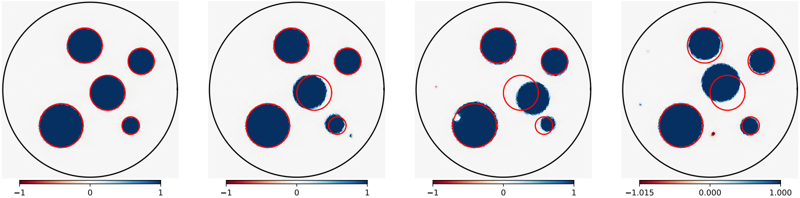
Reconstructing a star-shaped domain without noise : $N = 1, 5, 15.$



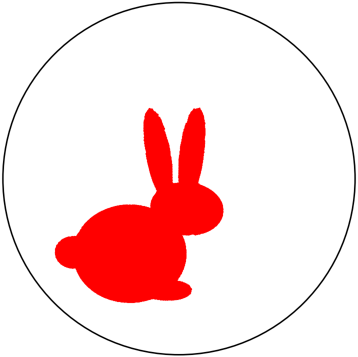
Reconstructing a star-shaped domain with noisy data $(N = 15)$: 1%, 3%, 5%.

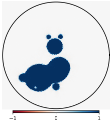
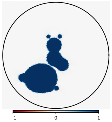
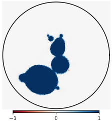
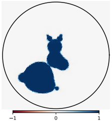
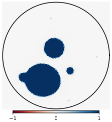
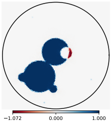
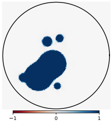
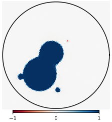
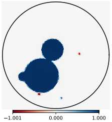
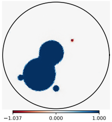


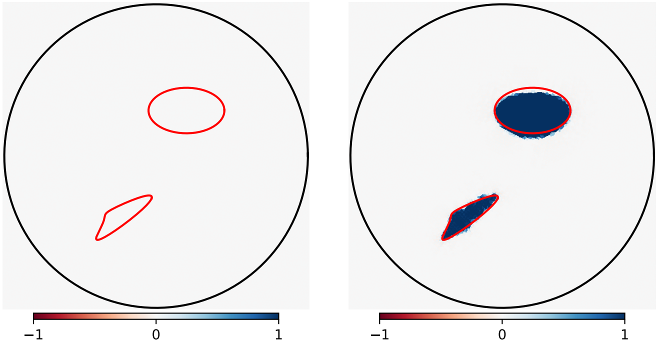
Reconstructing a domain convex in one direction with noisy data $(N = 12)$: 0%, 1%, 2%, 3%.



Reconstruction of disks with noisy data $(N = 10)$: 0%, 5%, 8%, 10%.



	$N = 10$	$N = 15$	$N = 20$	$N = 25$
0 %				
1 %				
3 %				
5 %				



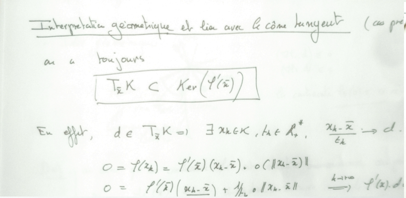
Instabilities may occur : false reconstruction with $N = 31$ (all the weights are almost zero) and correct one for $N = 32$.

Conclusion

- We presented a reconstruction method based on an original combination of Prony's method and quadrature domains :
  A reconstruction method for the inverse gravimetric problem
  Anthony Gerber-Roth, Alexandre Munnier and Karim Ramdani
  SMAI Journal of Computational Mathematics (2023)
  https://smai-jcm.centre-mersenne.org/articles/10.5802/smai-jcm.99/
- What's next ?
  - Surface uniform mass distribution.
  - Non uniform mass distribution.
  - 3D
  - Stokes problem (Alexandre Munnier).

Selected bibliography

- *"Quadrature domains"* Makoto Sakai (1982)
- *"Inverse source problems"* Victor Isakov (1990)
- *"Properties of some balayage operators, with applications to quadrature domains and moving boundary problems"* Björn Gustafsson and Makoto Sakai (1994)
- *"A stable numerical method for inverting shape from moments"* Gene H. Golub,Peyman Milanfar and James Varah (1999)
- *"Lectures on balayage"* Björn Gustafsson (2004)
- *"What is a quadrature domain ?"* Björn Gustafsson and Harold S. Shapiro (2005)
- *"On the accuracy of solving confluent Prony systems"* Dmitry Batenkov and Yosef Yomdin (2013)



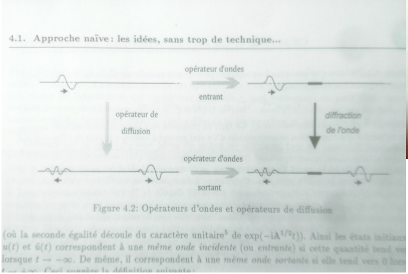
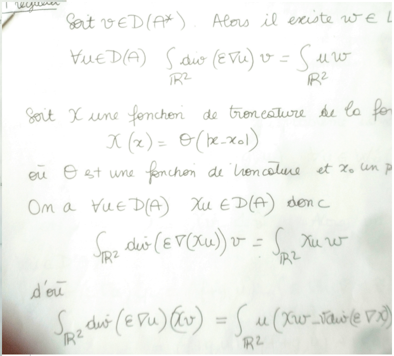
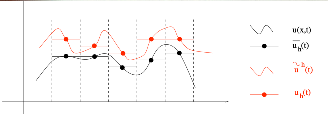

4.1. Approche naïve : les idées, sans trop de technique...

Figure 4.2: Opérateurs d'ondes et opérateurs de diffusion

(où la seconde égalité découle du caractère unitaire[5] de $\exp(-iA^{1/2}t)$). Ainsi les états initiaux $u(t)$ et $\tilde{u}(t)$ correspondent à une même onde incidente (ou entrante) si cette quantité tend vers 0 lorsque $t \to -\infty$. De même, il correspondent à une même onde sortante si elle tend vers 0 lorsque $t \to +\infty$. Ceci suggère la définition suivante :





...MERCI....

Further comments

Solving Prony's system

$$P(Z) := \prod_{n=1}^{N} (Z - z_n) = Z^N + \sum_{n=0}^{N-1} \alpha_n Z^n.$$

$$\left\{ \begin{array}{ccccccccc} c_1 & + & c_2 & + & \dots & + & c_N & = & \tau_0 \\ c_1 z_1 & + & c_2 z_2 & + & \dots & + & c_N z_N & = & \tau_1 \\ c_1 z_1^2 & + & c_2 z_2^2 & + & \dots & + & c_N z_N^2 & = & \tau_2 \\ & & & & \vdots & & & = & \vdots \\ c_1 z_1^{N-1} & + & c_2 z_2^{N-1} & + & \dots & + & c_N z_N^{N-1} & = & \tau_{N-1} \\ c_1 z_1^{N} & + & c_2 z_2^{N} & + & \dots & + & c_N z_N^{N} & = & \tau_{N} \\ & & & & \vdots & & & = & \vdots \\ c_1 z_1^{2N-1} & + & c_2 z_2^{2N-1} & + & \dots & + & c_N z_N^{2N-1} & = & \tau_{2N-1} \end{array} \right.$$

Solving Prony's system

$$P(Z) := \prod_{n=1}^{N} (Z - z_n) = Z^N + \sum_{n=0}^{N-1} \alpha_n Z^n.$$

$$\left\{\begin{array}{ccccccccccc}
c_1 & + & c_2 & + & \dots & + & c_N & = & \tau_0 & \times\alpha_0 \\
c_1 z_1 & + & c_2 z_2 & + & \dots & + & c_N z_N & = & \tau_1 & \times\alpha_1 \\
c_1 z_1^2 & + & c_2 z_2^2 & + & \dots & + & c_N z_N^2 & = & \tau_2 & \times\alpha_2 \\
 & & & & \vdots & & & = & \vdots & \\
c_1 z_1^{n-1} & + & c_2 z_2^{n-1} & + & \dots & + & c_N z_N^{N-1} & = & \tau_{N-1} & \times\alpha_{N-1} \\
c_1 z_1^{n} & + & c_2 z_2^{n} & + & \dots & + & c_N z_N^{N} & = & \tau_{N} & \times 1 \\
 & & & & \vdots & & & = & \vdots & \\
c_1 z_1^{2N-1} & + & c_2 z_2^{2N-1} & + & \dots & + & c_N z_N^{2N-1} & = & \tau_{2N-1} &
\end{array}\right.$$

$$c_1 P(z_1) + c_2 P(z_2) + \cdots + c_N P(z_N) =$$

$$\alpha_0 \tau_0 + \alpha_1 \tau_1 + \cdots + \cdots + \alpha_{N-1} \tau_{N-1} + \tau_N$$

$$\alpha_0 \tau_0 + \alpha_1 \tau_1 + \cdots + \cdots + \alpha_{n-1} \tau_{n-1} = -\tau_n.$$

Solving Prony's system

$$P(Z) := \prod_{n=1}^{N}(Z - z_n) = Z^N + \sum_{n=0}^{N-1} \alpha_n Z^n.$$

$$\left\{\begin{array}{ccccccccccl}
c_1 & + & c_2 & + & \dots & + & c_N & = & \tau_0 & & \\
c_1 z_1 & + & c_2 z_2 & + & \dots & + & c_N z_N & = & \tau_1 & & \times\alpha_0 \\
c_1 z_1^2 & + & c_2 z_2^2 & + & \dots & + & c_N z_N^2 & = & \tau_2 & & \times\alpha_1 \\
 & & & & \vdots & & & = & \vdots & & \\
c_1 z_1^{N-1} & + & c_2 z_2^{N-1} & + & \dots & + & c_N z_N^{N-1} & = & \tau_{N-1} & & \times\alpha_{N-2} \\
c_1 z_1^{N} & + & c_2 z_2^{N} & + & \dots & + & c_N z_N^{N} & = & \tau_{N} & & \times\alpha_{N-1} \\
c_1 z_1^{N+1} & + & c_2 z_2^{N+1} & + & \dots & + & c_N z_N^{N+1} & = & \tau_{N+1} & & \times 1 \\
 & & & & \vdots & & & = & \vdots & & \\
c_1 z_1^{2N-1} & + & c_2 z_2^{2N-1} & + & \dots & + & c_N z_N^{2N-1} & = & \tau_{2N-1} & &
\end{array}\right.$$

$$c_1 z_1 P(z_1) + \dots + c_n z_n P(z_n) = \alpha_0\tau_1 + \alpha_1\tau_2 + \dots + \dots + \alpha_{n-1}\tau_n + \tau_{n+1}$$

$$\alpha_0\tau_1 + \alpha_1\tau_2 + \dots + \dots + \alpha_{n-1}\tau_n = -\tau_{n+1}.$$

Solving Prony's system

$$\begin{pmatrix} \tau_0 & \tau_1 & \cdots & \tau_{n-1} \\ \tau_1 & \tau_2 & \cdots & \tau_n \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{n-1} & \tau_n & \cdots & \tau_{2n-2} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = - \begin{pmatrix} \tau_n \\ \tau_{n+1} \\ \vdots \\ \tau_{2n-1} \end{pmatrix} \iff \mathbb{T}_0 \boldsymbol{\alpha} = -\boldsymbol{\tau}'.$$