A short overview of sampling methods in waveguides: “une histoire de mode”

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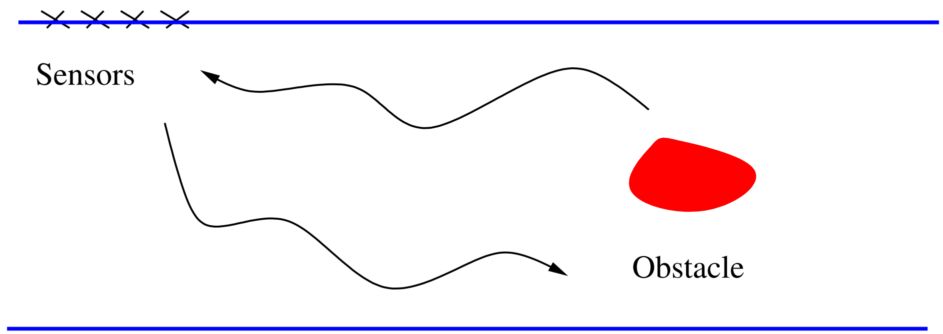
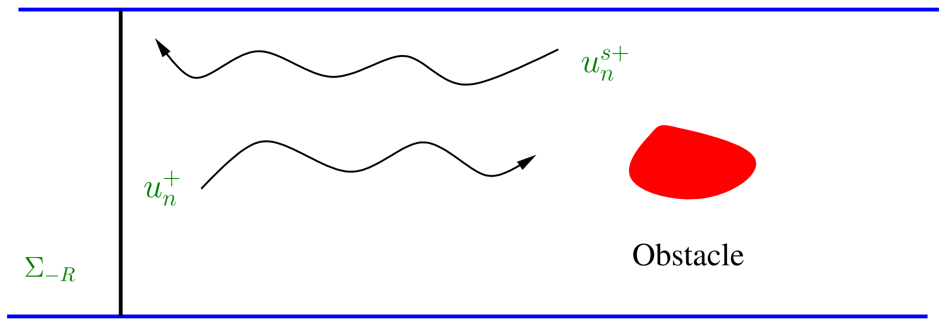
Objective: sending incident waves and measuring the corresponding scattered waves to find obstacles

Real problem: surface data in the time domain



Treated problem: modal data in the frequency domain



The **total fields** $u = u^s + u^i$ satisfy (for ex. $u^i = u_n^+$ for $n \in \mathbb{N}$):

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega \setminus \overline{O} \\ \partial_\nu u = 0 & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial O \\ u^s & \text{is outgoing} \end{cases}$$

Computing the modes

- The modes : find u with separate variables (x,y) such that

$$\begin{cases} (\Delta + k^2)u = 0 & \text{in} \quad \Omega \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

- φ_n and μ_n $(n \in \mathbb{N})$: Neumann eigenfunctions and eigenvalues of the $1D$ operator $-\Delta$ in transverse section Σ

- The φ_n form a complete basis of $L^2(\Sigma)$ while $\mu_0 = 0 < \mu_1 < \mu_2 < \mu_3 < \cdots < \mu_n \longrightarrow +\infty$

- Assumption on k : suppose $k \neq \sqrt{\mu_n}$ for all $n \in \mathbb{N}$

- The modes : $u_n^\pm(x,y) = e^{\pm\lambda_n x}\varphi_n(y)$, $\lambda_n = i\sqrt{k^2 - \mu_n}$ for $n = 0, \cdots, P-1$ (propagating modes) and $\lambda_n = -\sqrt{\mu_n - k^2}$ for $n \geq P$ (evanescent modes)

The fundamental solution

For $M' \in \Omega$, the solution to

$$\begin{cases} -(\Delta + k^2)G(\cdot, M') = \delta_{M'} & \text{in} \quad \Omega \\ \partial_\nu G(\cdot, M') = 0 & \text{on} \quad \partial\Omega \\ G(\cdot, M') \quad \text{is outgoing} & \end{cases}$$

is given by

$$G(M, M') = -\sum_{n \in \mathbb{N}} \frac{e^{\lambda_n |x - x'|}}{2\lambda_n} \varphi_n(y) \varphi_n(y')$$

A modal decomposition: for $x > x'$,

$$G(M, M') = -\sum_n \frac{1}{2\lambda_n} e^{\lambda_n x} \varphi_n(y) e^{-\lambda_n x'} \varphi_n(y') = -\sum_n \frac{1}{2\lambda_n} u_n^+(M) u_n^-(M')$$

$\longrightarrow$ if u_n^{s+} is the scattered field for $u^i = u_n^+$, the scattered field $u^s(\cdot, M')$ for $u^i = G(\cdot, M')$ is given by $u^s(M, M') = -\sum_n \frac{1}{2\lambda_n} u_n^{s+}(M) u_n^-(M')$

The Linear Sampling Method

The near-field operator :

$$\begin{cases} \mathcal{N} : L^2(\Sigma_{-R}) \to L^2(\Sigma_{-R}) \\ \quad h \mapsto \mathcal{N}h, \quad (\mathcal{N}h)(M) = \displaystyle\int_{\Sigma_{-R}} u^s(M, M')h(M')\, ds(M') \quad M \in \Sigma_{-R}, \end{cases}$$

where $u^s(\cdot, M')$ is the scattered field for $u^i = G(\cdot, M')$

Property: for $Z \in \Omega$, we have $G(\cdot, Z)|_{\Sigma_{-R}} \in \text{Range}(\mathcal{N})$ "$\Longleftrightarrow$" $Z \in O$

Principle:

- solve equation $\mathcal{N}h = G(\cdot, Z)|_{\Sigma_{-R}}$ in $L^2(\Sigma_{-R})$ for all Z
- plot $\psi(Z) = 1/\|h(Z)\|_{L^2(\Sigma_{-R})}$: indicator function of the defect
- obtain an "image" of the defect

The Linear Sampling Method (discretization)

Modal decomposition:

$$G(M,M') = -\sum_n \frac{1}{2\lambda_n} u_n^+(M) u_n^-(M'), \quad u^s(M,M') = -\sum_n \frac{1}{2\lambda_n} u_n^{s+}(M) u_n^-(M')$$

Projection on Σ_{-R} :

$$u_n^{s+}|_{\Sigma_{-R}} = \sum_{m\in\mathbb{N}} S_{mn}^{+-}\,\varphi_m, \quad h = \sum_{n\in\mathbb{N}} h_n^-\,\varphi_n$$

Near-field equation $\mathcal{N}h = G(\cdot, Z)|_{\Sigma_{-R}}$ in $L^2(\Sigma_{-R})$ is equivalent to

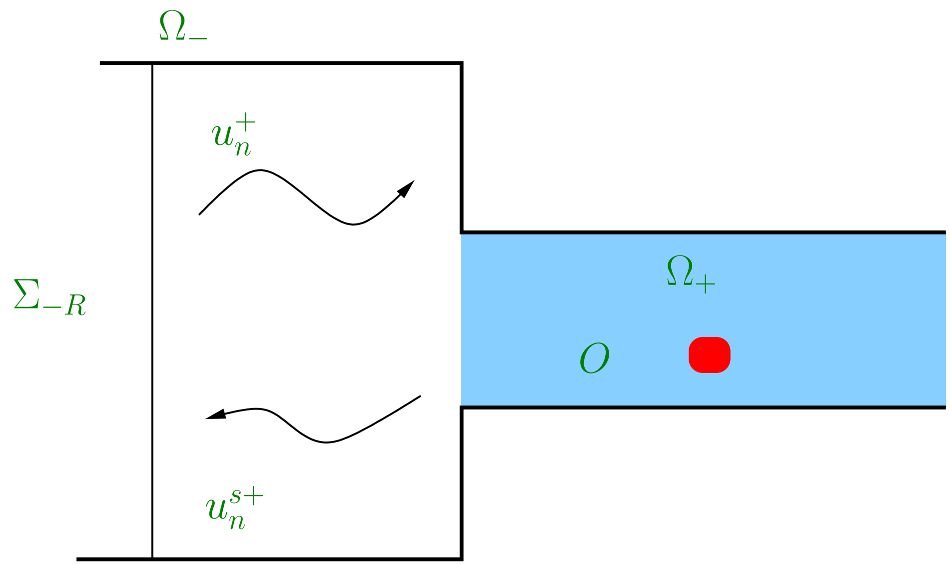
$$\forall m \in \mathbb{N}, \quad \sum_{n\in\mathbb{N}} \frac{e^{\lambda_n R}}{\lambda_n} S_{mn}^{+-}\, h_n^- = \frac{e^{\lambda_m R}}{\lambda_m} u_m^+(Z)$$

Discretization (far-field approximation): we restrict to $m, n = 0, \cdots, P-1$, where P is the number of propagating modes

How general could be this approach ?

The junction of waveguides

Two half-waveguides
having different properties:
the fundamental solution G
is not simple any more



Reference field r_n^+ for all $n \in \mathbb{N}$: diffracted solution associated with u_n^+ in the presence of the junction only (no obstacle)

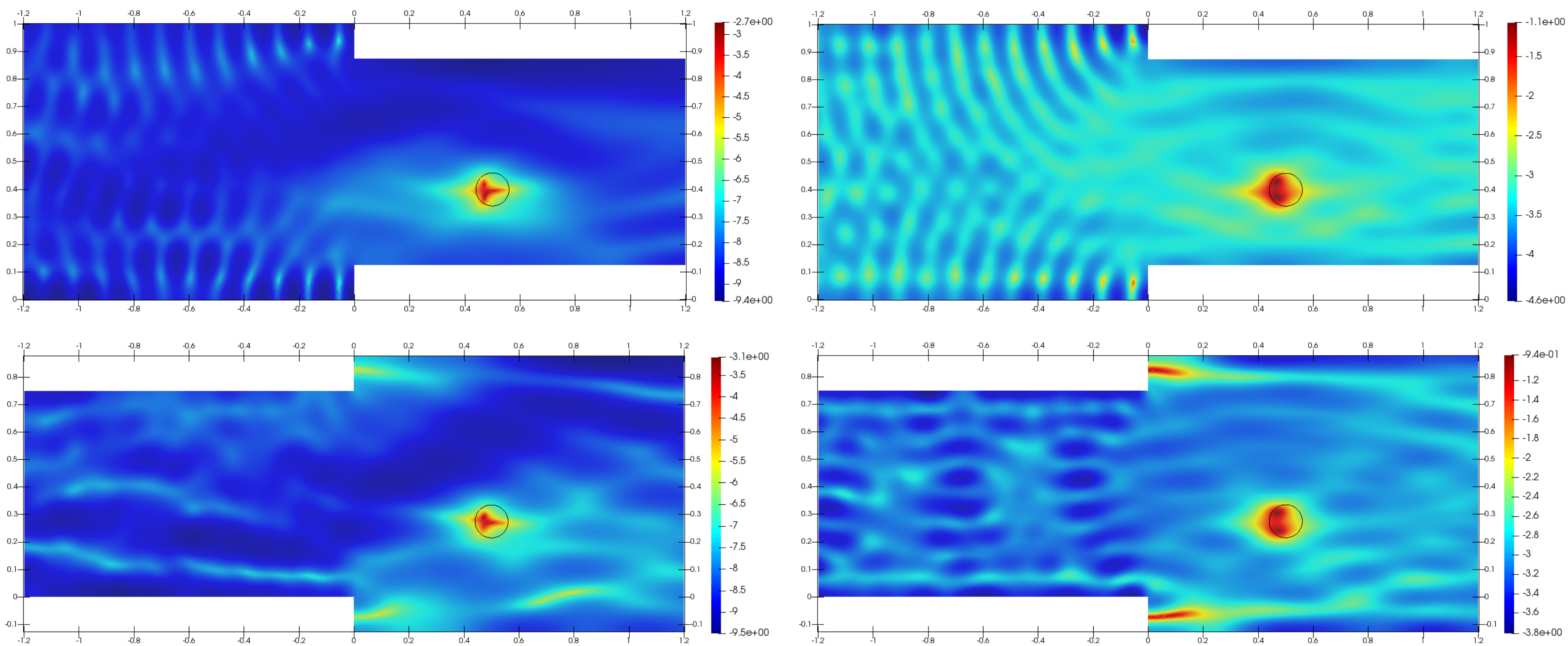
Near-field equation $\mathcal{N}h = G(\cdot, Z)|_{\Sigma_{-R}}$ in $L^2(\Sigma_{-R})$ is equivalent to

$$\forall m \in \mathbb{N}, \quad \sum_{n \in \mathbb{N}} \frac{e^{\lambda_n R}}{\lambda_n} S_{mn}^{+-} \, h_n^- = \frac{e^{\lambda_m R}}{\lambda_m} r_m^+(Z)$$

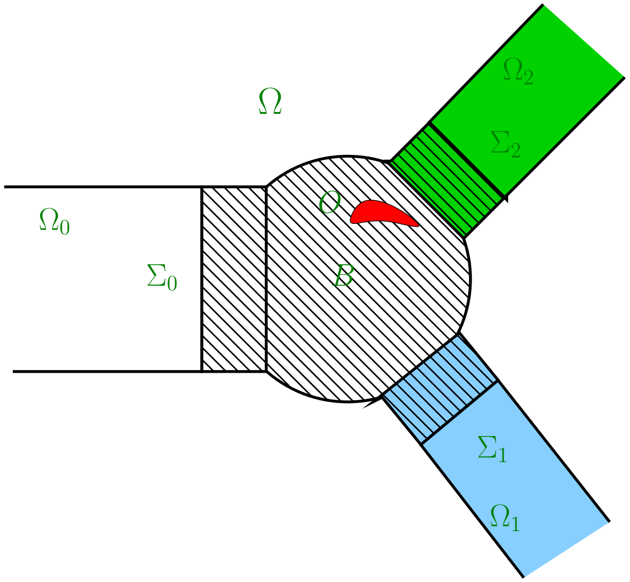
Remark: The reference fields r_n^+ are pre-computed with a Finite Element Method

Numerical experiments

- Number of propagating modes : $P = 10$ (large waveguide) and $\tilde{P} = 8$ (small waveguide)
- Exact data (left picture) and noisy data of amplitude 10% (right picture)

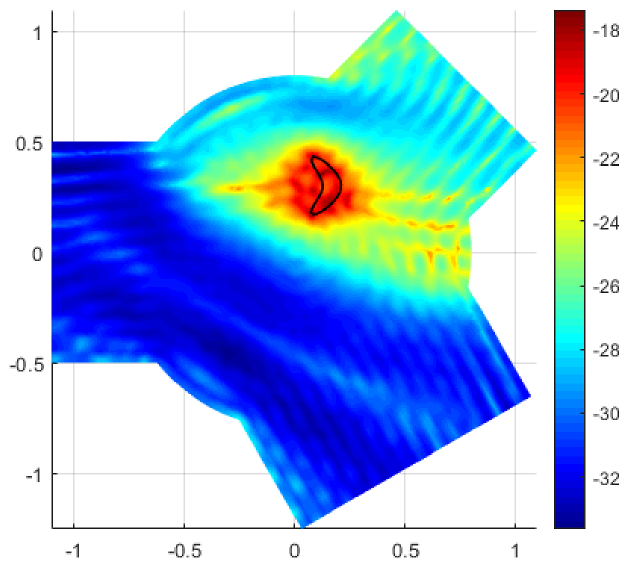


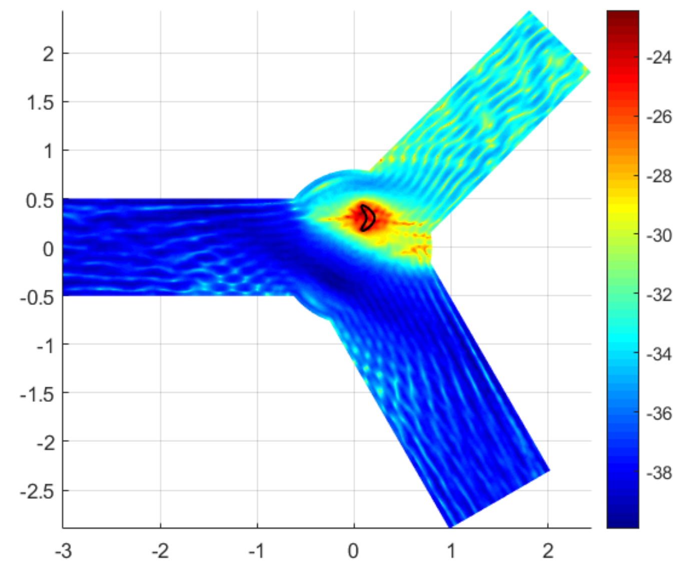
Numerics : junction of three waveguides



Number of propagating modes :
$P(0) = 13$, $P(1) = 16$ and $P(2) = 12$

Data on **close** Σ_0 (left) and **far** Σ_0 (right)



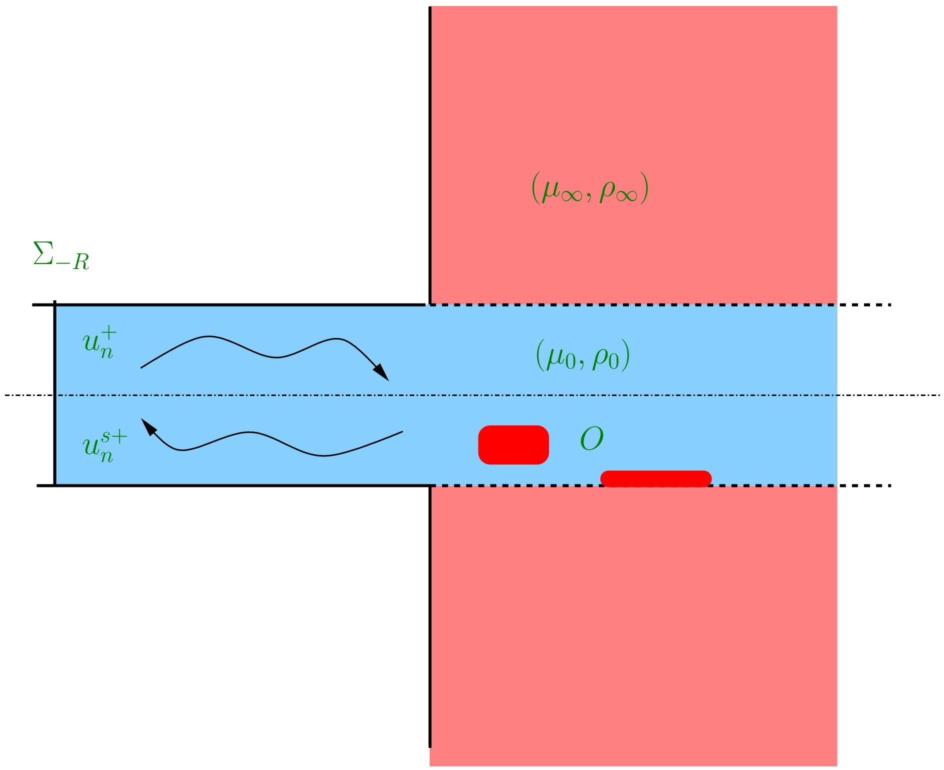


A closed waveguide embedded in an infinite medium

Objective : finding obstacles from scattering measurements using modes coming from the free waveguide

A special case :

$$c_0 := \sqrt{\frac{\mu_0}{\rho_0}} > c_\infty := \sqrt{\frac{\mu_\infty}{\rho_\infty}}$$



Applications : a steel cable in concrete (civil engineering), a metallic tube in a liquid (oil/gas industry)

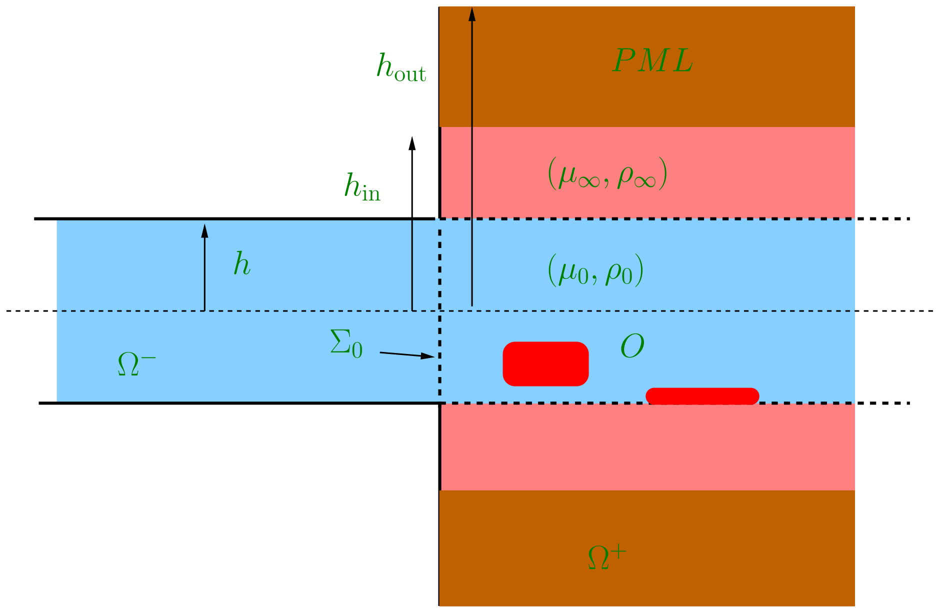
Both the forward and the inverse problems are difficult !

Using transverse PMLs

We introduce $\Omega^- = (-\infty, 0) \times (-h, h)$, $\Omega^+ = (0, +\infty) \times (-h_{\text{out}}, h_{\text{out}})$ and $\Omega = \Omega^- \cup \Sigma_0 \cup \Omega^+$, with $\Sigma_0 := \{0\} \times (-h, h)$

Problem : find u such that

$$\begin{cases} Pu = 0 & \text{in } \Omega \setminus \overline{O} \\ \partial_\nu u = 0 & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial O \\ u - u^i & \text{is outgoing} \end{cases}$$

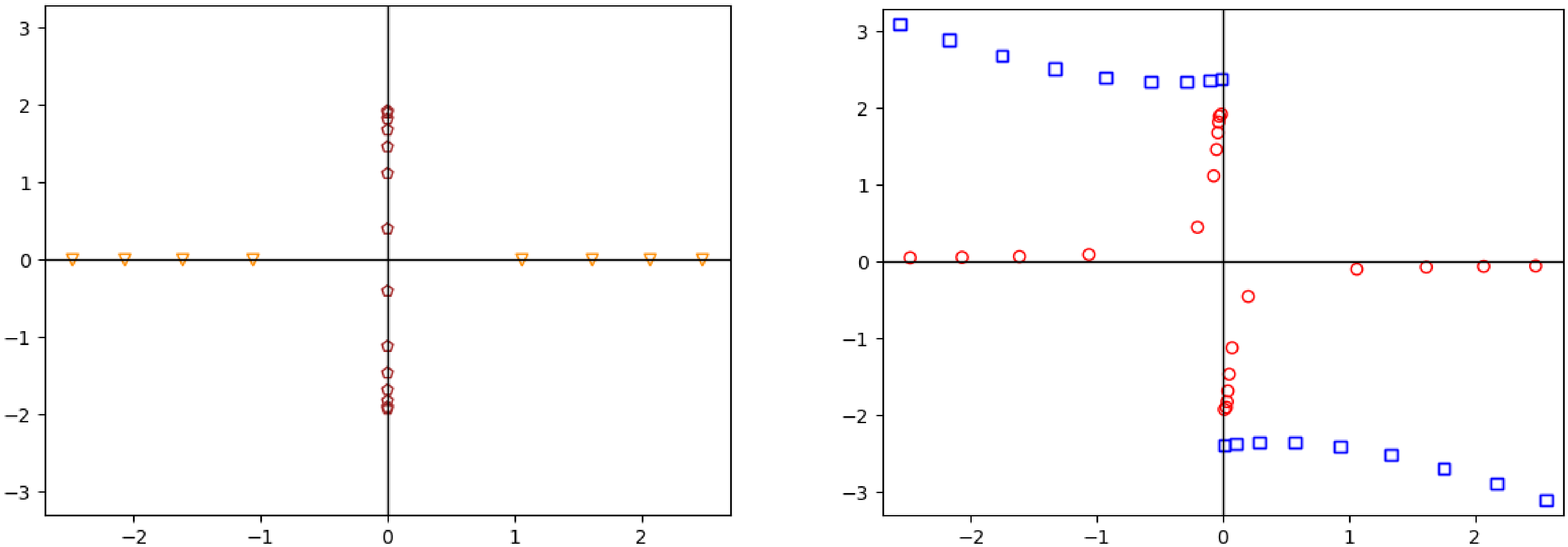


with

$$P = -\partial_y(\alpha\mu\,\partial_y\,\cdot) - \frac{\mu}{\alpha}\partial_{xx}\,\cdot - \frac{\mu}{\alpha}k^2$$

- $(\alpha, \mu, k) = (1, \mu_0, k_0)$ in the blue zone
- $(\alpha, \mu, k) = (1, \mu_\infty, k_\infty)$ in the pink zone
- $(\alpha, \mu, k) = (\alpha_\infty, \mu_\infty, k_\infty)$ in the PML : $\alpha_\infty \in \mathbb{C}$, $\arg(\alpha_\infty) \in (-\frac{\pi}{2}, 0)$

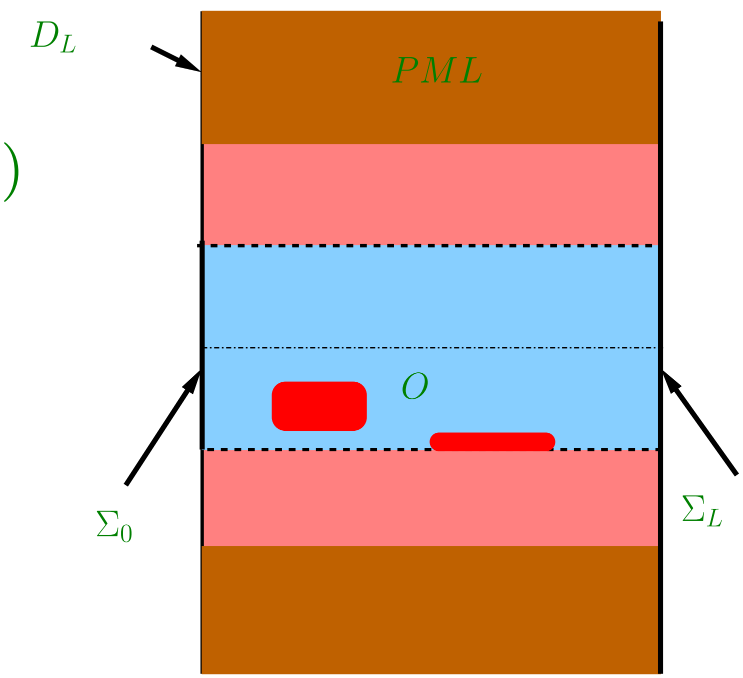
The spectrum in the free/stratified waveguide



- **Left**: **free** waveguide (cardinal red pentagons : propagating modes, yellow triangles : evanescent modes)
- **Right**: **embedded** waveguide closed by truncated abrupt PMLs (red circles : leaky modes, blue squares : PML modes)

Using longitudinal DtN operators

$$\left\{\begin{array}{rcl} Pu &=& 0 \text{ in } D_L \\ \partial_\nu u &=& 0 \text{ on } \partial D_L \setminus (\Sigma_0 \cup \Sigma_L \cup \partial O) \\ u &=& 0 \text{ on } \partial O \\ -\mu_0 \partial_x u &=& T_0 u - 2\mu_0 \partial_x u^i \text{ on } \Sigma_0 \\ \frac{\mu}{\alpha}\partial_x u &=& T_L u \text{ on } \Sigma_L \end{array}\right.$$



where

$$P = -\partial_y(\alpha\mu\,\partial_y\,\cdot) - \frac{\mu}{\alpha}\partial_{xx}\cdot - \frac{\mu}{\alpha}k^2$$

DtN operators on Σ_0 and Σ_L:

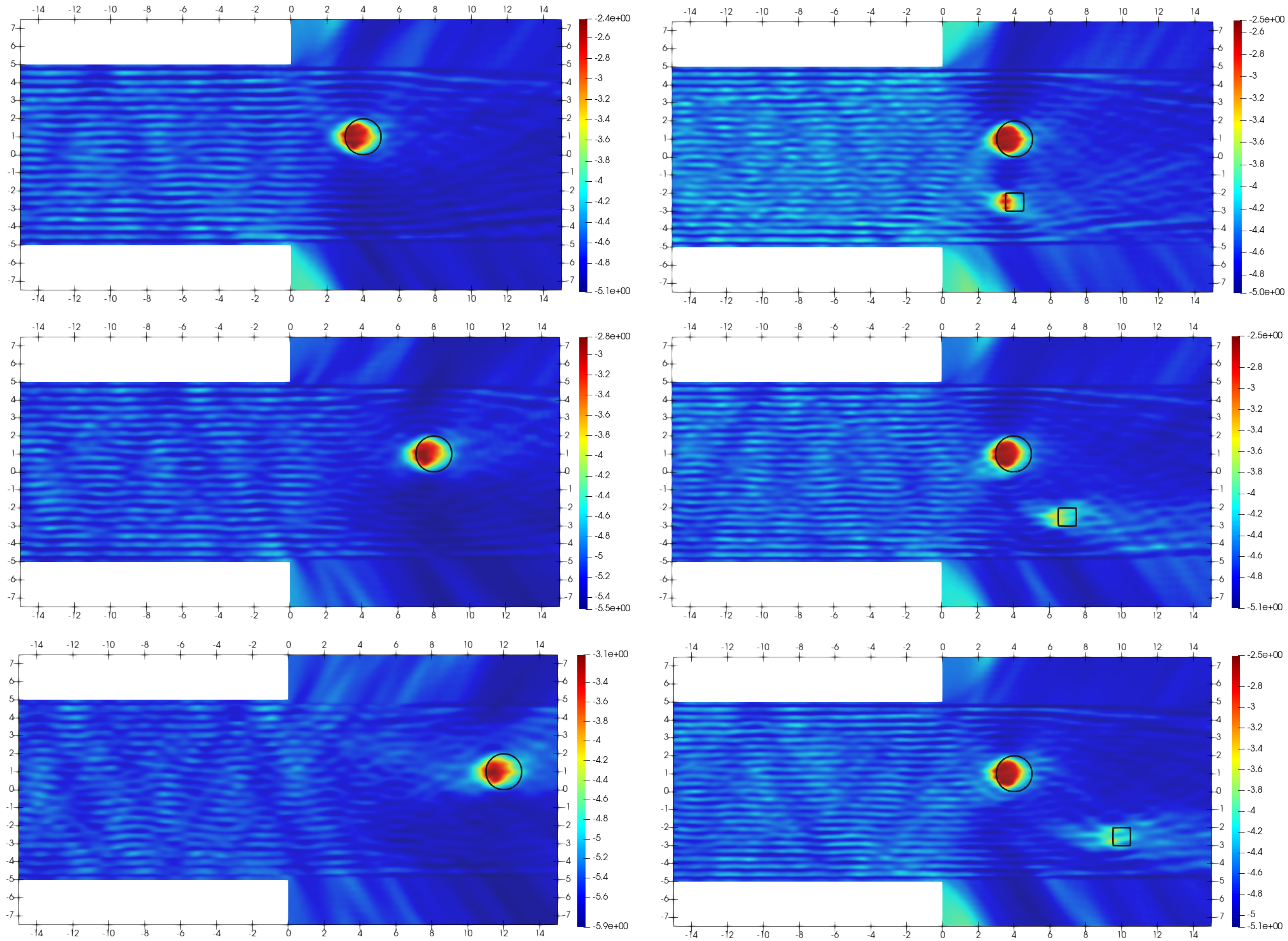
$$T_0 \;:\; \varphi \mapsto -\mu_0 \partial_x u^-(\varphi)|_{\Sigma_0} = \mu_0 \sum_{n\in\mathbb{N}} \lambda_n (\varphi, \varphi_n)_{L^2(\Sigma_0)} \varphi_n$$

$$T_L \;:\; \varphi \mapsto \frac{\mu}{\alpha}\partial_x u^+(\varphi)|_{\Sigma_L}$$ (well-defined from unperturbed case)

$\longrightarrow$ An explicit expression of T_L is unknown !

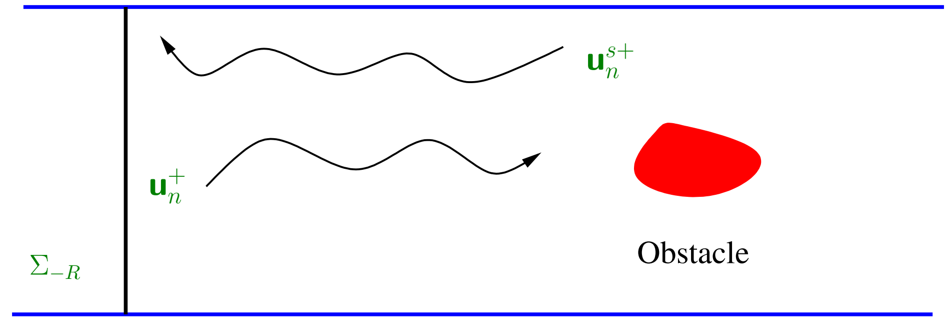
Numerical experiments

Influence of the distance obstacle/interface:



Extension to elasticity

$$\begin{cases} \sigma(\mathbf{u}) = \lambda \mathrm{tr}(e(\mathbf{u}))\,\mathrm{Id} + 2\mu\, e(\mathbf{u}) \\ e(\mathbf{u}) = (\nabla \mathbf{u} + {}^T\nabla \mathbf{u})/2 \end{cases}$$



The **total fields** $\mathbf{u} = \mathbf{u}^s + \mathbf{u}^i$ satisfy (for ex. $\mathbf{u}^i = \mathbf{u}_n^+$ for $n \in \mathbb{N}$):

$$\begin{cases} \mathrm{div}\sigma(\mathbf{u}) + \rho\omega^2 \mathbf{u} = 0 & \text{in } \Omega \setminus \overline{O} \\ \sigma(\mathbf{u}) \cdot \boldsymbol{\nu} = 0 & \text{on } \partial\Omega \\ \mathbf{u} = 0 & \text{on } \partial O \\ \mathbf{u}^s & \text{is outgoing} \end{cases}$$

Guided modes : solutions in the form $\mathbf{u}(x, y) = e^{\lambda x}\boldsymbol{u}(y)$ to problem

$$\begin{cases} \mathrm{div}\sigma(\mathbf{u}) + \rho\omega^2 \mathbf{u} = 0 & \text{in } \Omega \\ \sigma(\mathbf{u}) \cdot \boldsymbol{\nu} = 0 & \text{on } \partial\Omega \end{cases}$$

Guided modes in elastodynamics

- Hybrid variables $(\mathbf{X}, \mathbf{Y})$:

$$\mathbf{u} = \begin{pmatrix} \mathsf{u}_x \\ \mathsf{u}_y \end{pmatrix} \quad \sigma(\mathbf{u}) \cdot e_x = \begin{pmatrix} -\mathsf{t}_x \\ \mathsf{t}_y \end{pmatrix} \rightarrow \mathbf{X} = \begin{pmatrix} \mathsf{t}_y \\ \mathsf{u}_x \end{pmatrix} \quad \mathbf{Y} = \begin{pmatrix} \mathsf{u}_y \\ \mathsf{t}_x \end{pmatrix}$$

- Reformulation of elastodynamics :

$$\frac{\partial}{\partial x} \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} = \begin{pmatrix} 0 & F_Y \\ F_X & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}$$

- Guided modes :

$$\begin{pmatrix} \mathbf{X}_n^{\pm} \\ \mathbf{Y}_n^{\pm} \end{pmatrix} = \begin{pmatrix} \pm \boldsymbol{\mathcal{X}}_n(y) \\ \boldsymbol{\mathcal{Y}}_n(y) \end{pmatrix} e^{\pm \lambda_n x}, \quad \mathbf{U}_n^{\pm} = \begin{pmatrix} u_y^n(y) \\ \pm u_x^n(y) \end{pmatrix} e^{\pm \lambda_n x}$$

- Bi-orthogonality and completeness :

$$(\boldsymbol{\mathcal{X}}_n | \boldsymbol{\mathcal{Y}}_m)_\Sigma = (u_y^n, t_y^m)_\Sigma + (u_x^n, t_x^m)_\Sigma = \delta_{nm},$$

$$\boldsymbol{\mathcal{X}} = \sum_{n>0} (\boldsymbol{\mathcal{X}} | \boldsymbol{\mathcal{Y}}_n)_\Sigma \boldsymbol{\mathcal{X}}_n, \quad \boldsymbol{\mathcal{Y}} = \sum_{n>0} (\boldsymbol{\mathcal{X}}_n | \boldsymbol{\mathcal{Y}})_\Sigma \boldsymbol{\mathcal{Y}}_n$$

Extended Green function

- The extended Green function $G(\cdot, M')$ satisfies

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial x} G(\cdot, M') = \begin{pmatrix} 0 & F_Y \\ F_X & 0 \end{pmatrix} G(\cdot, M') - \delta_{M'} \begin{pmatrix} \mathrm{Id}_2 & 0_2 \\ 0_2 & \mathrm{Id}_2 \end{pmatrix} \text{ in } \Omega_R \\ \sigma_{yy}(G(\cdot, M')) = 0 \quad \mathtt{t}_y(G(\cdot, M')) = 0 \quad \text{on} \quad \partial\Omega_R \\ T_\pm G_Y(\cdot, M') = \pm G_X(\cdot, M') \quad \text{on} \quad \Sigma_{\pm R}, \end{array}\right.$$

- A 4×4 matrix :

$$G(M, M') = -\sum_n \begin{pmatrix} s(x - x')\boldsymbol{\mathcal{X}}_n(y)\,{}^T\boldsymbol{\mathcal{Y}}_n(y') & \boldsymbol{\mathcal{X}}_n(y)\,{}^T\boldsymbol{\mathcal{X}}_n(y') \\ \boldsymbol{\mathcal{Y}}_n(y)\,{}^T\boldsymbol{\mathcal{Y}}_n(y') & s(x - x')\boldsymbol{\mathcal{Y}}_n(y)\,{}^T\boldsymbol{\mathcal{X}}_n(y') \end{pmatrix} \frac{e^{\lambda_n |x - x'|}}{2}$$

- Symmetry relationships :

$$G_u^\sigma(M, M') = {}^T G_u^\sigma(M', M), \quad G_u^Y(M, M') = -{}^T G_X^\sigma(M', M)$$

The Linear Sampling Method again

Using the $(\mathbf{X},\mathbf{Y})$ variables : for $M' \in \Sigma_{-R}$, the scattered field $u^s_Y(\cdot, M')$ is associated with $u^i = G^Y_u(\cdot, M')$ (G : extended Green function)

The near-field operator :

$$\begin{cases} \mathcal{N} : \left(L^2(\Sigma_{-R})\right)^2 \to \left(L^2(\Sigma_{-R})\right)^2 \\ \qquad \mathbf{h} \mapsto \mathcal{N}\mathbf{h}, \quad (\mathcal{N}\mathbf{h})(M) = \int_{\Sigma_{-R}} X^s_Y(M, M') \cdot \mathbf{h}(y)\, ds(M'), \quad M \in \Sigma_{-R} \end{cases}$$

Near-field equation : for all $Z \in \Omega$, solve in $\left(L^2(\Sigma_{-R})\right)^2$

$$\mathcal{N}\mathbf{h} = G^Y_X(\cdot, Z)|_{\Sigma_{-R}} \cdot \mathbf{p}$$

by using the modal projection ($\mathbf{p}$: polarization vector of $\mathbb{R}^2$)

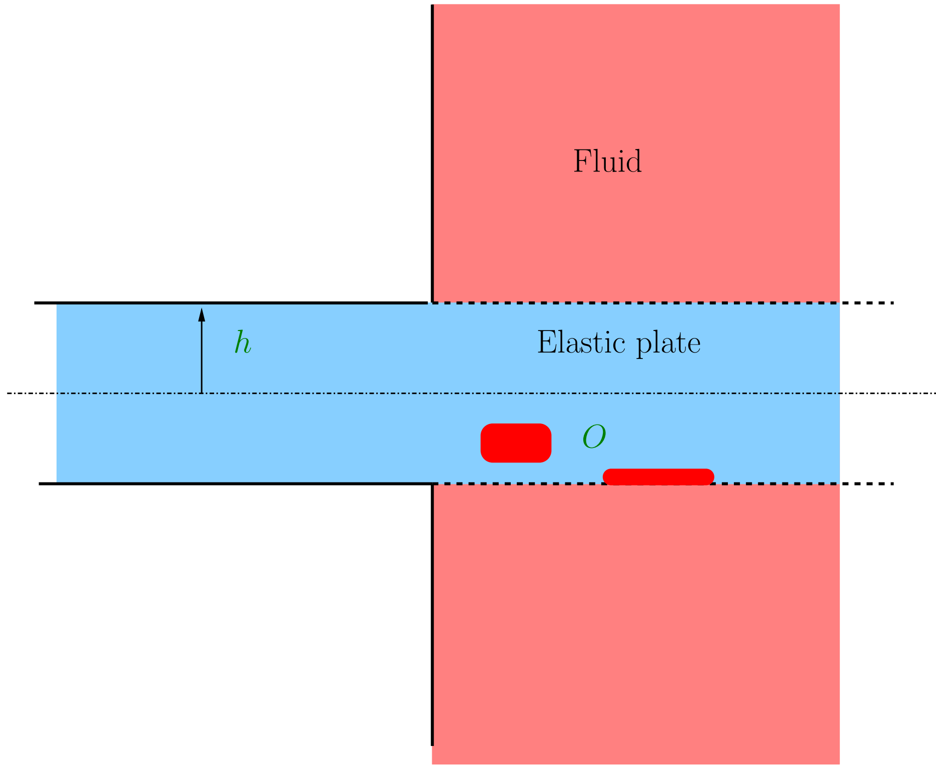
Imaging the defect : plot $\psi(Z) = 1/\|\mathbf{h}(Z)\|_{\left(L^2(\Sigma_{-R})\right)^2}$

Extension a to solid/fluid interaction problem

Objective : scattering in an elastic plate which is partially immersed in a fluid

Displacement/velocity potential (u, φ) satisfies

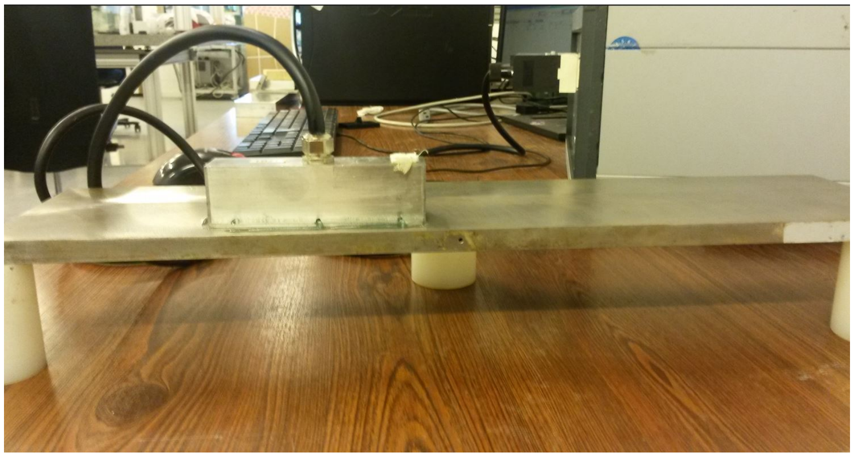
$$\left\{\begin{array}{rl} \mathrm{div}(\sigma(\mathbf{u})) + \omega^2 \rho_s \mathbf{u} = 0 & \text{solid} \\ \Delta\varphi + k_f^2 \varphi = 0 & \text{fluid} \\ \sigma(\mathbf{u}) \cdot n_s = -i\omega\rho_f\, \varphi n_s & \text{interface} \\ -i\omega(\mathbf{u} \cdot n_s) = \nabla\varphi \cdot n_s & \text{interface} \\ \mathbf{u} = 0 & \text{on } \partial O \\ \mathrm{BC} + \mathrm{RC} & \end{array}\right.$$

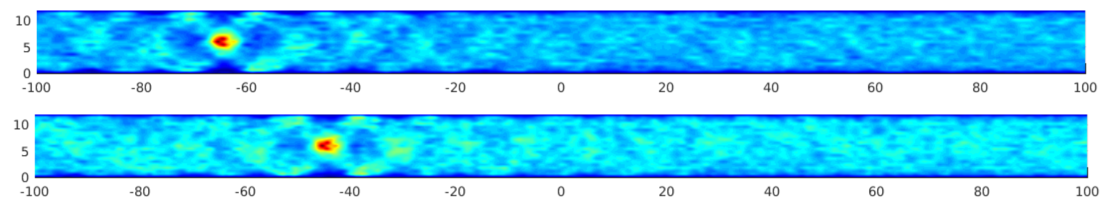


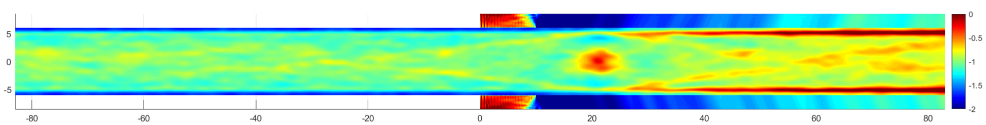
$\longrightarrow$ We again use PMLs in the transverse direction

Real experiments at CEA/LIST

We apply the LMS to real surface data in the time domain







Thank you very much for your attention !