Journées Ondes des Poètes 2024

Wave propagation in the presence of negative materials: a focus on Anne-Sophie's work

Mahran Rihani

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Ensta Paris, April 19, 2024

First contact and key words

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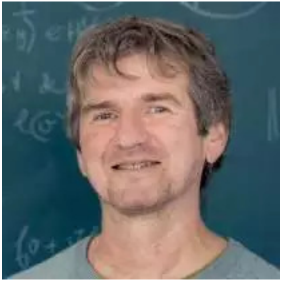
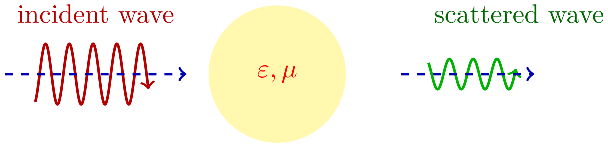


First contact and key words

Introduction: what is a negative material?

- The scattering of (harmonic) electromagnetic waves by an obstacle of permittivity ε and of permeability μ.



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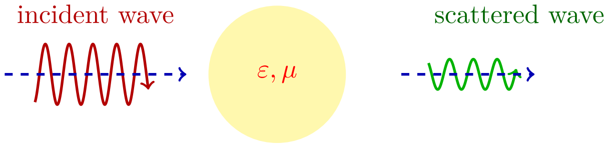

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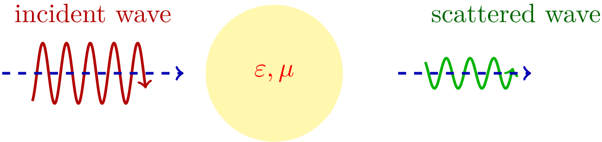
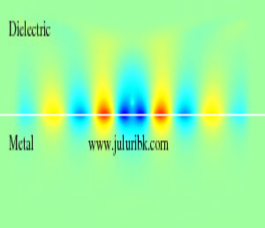


- Classically, we suppose that ε and μ are real positive constants.
- In real applications, almost all media are dissipative and dispersive: ε and μ are, usually, complex functions of ω.
- At optical frequencies, some metals have a dielectric permittivity ε with negative real part and very small imaginary part.
- Some meta-materials have, in some range of frequencies, negative ε and/or negative μ.

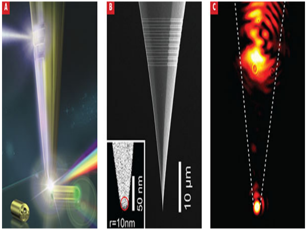
Negative materials= some metals or some meta-materials.

Applications and unusual phenomena

- Combing negative materials with positive ones leads to many new applications and unusual phenomena:



Surface Plasmon Polariton.



Plasmonic nanofocusing spectroscopy.

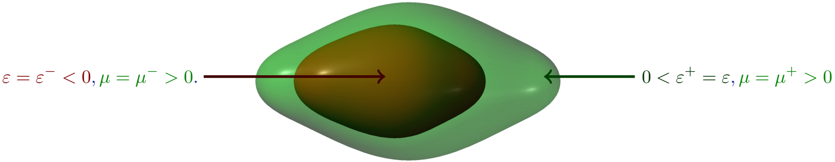
Lycurgus cup (4th century):

Localized Surface Plasmon.

- We need to associate negative materials with positive ones.
- The shape of the interface plays an important role.

Introduction: modeling of the problem

- In the time-harmonic regime ($e^{-i\omega t}$), two models can be considered



The scalar problem: Find $u \in \mathrm{V}$ s.t. $-\mathrm{div}\,(\varepsilon \nabla u) - \omega^2 \mu u = f.$

The Maxwell's problem: Find $\boldsymbol{E} \in \mathbf{X}$ s.t $\mathbf{curl}\, \mu^{-1} \mathbf{curl}\, \boldsymbol{E} - \omega^2 \varepsilon \boldsymbol{E} = i\omega \boldsymbol{J}.$

⚠ Since ε changes sign, these problems may be ill-posed (in the Fredholm sense).

- Two important questions arise:
  1. Under which condition on ε these problems are well-posed (in the Fredholm sense)?
  2. What happens when Fredholmness is lost? (More difficult). This corresponds to all the new interesting phenomena.

Introduction: a bit of history 1/2

- M. Costabel, E. Stephan, A direct boundary integral method for transmission problems, J. Math. Anal. Appl. 106 (1985) 367–413.
- P. Ola, Remarks on a transmission problem, J. Math. Anal. Appl. 196 (1993) 639–658.

Using integral equations:

- When interface is smooth and $\kappa_\varepsilon = \varepsilon^-/\varepsilon^+ \neq -1$, the scalar problem is of Fredholm type. The case of 2D polygonal interfaces is also considered.

Introduction: a bit of history 1/2

- M. Costabel, E. Stephan, A direct boundary integral method for transmission problems, J. Math. Anal. Appl. 106 (1985) 367–413.
- P. Ola, Remarks on a transmission problem, J. Math. Anal. Appl. 196 (1993) 639–658.
- K. Ramdani, Lignes supraconductrices: Analyse mathématique et numérique, Ph.D. Thesis, Université Paris 6, 1999.
    - A.-S. Bonnet-Ben Dhia, M. Dauge, K. Ramdani, Analyse spectrale et singularités d'un problème de transmission non coercif, C. R. Acad. Sci. Ser. I 328 (1999) 717–720.
    - A.-S. Bonnet-Ben Dhia, K. Ramdani, A non elliptic spectral problem related to the analysis of superconducting micro-strip lines, ESAIM, V 36, 461–487, 2002

Using integral equations:

- For lipshitz interfaces the scalar problem is of Fredholm type for $\kappa_\varepsilon \in (-\infty;0)\backslash I_\Sigma$, I_Σ is called the critical interval.

Introduction: a bit of history

- In the early 2000s, many researchers suggested new applications of negative materials.
- C.M. Zwölf, Méthodes variationnelles pour la modélisation des problèmes de transmission d'onde électromagnétique entre diélectrique et métamatériau, Ph.D. Thesis, Université de Versailles Saint Quentin en Yvelines, 2007.

- The introduction of the T-coercivity and many others results.
- Progress about the numerical approximation of the problems.

Introduction: a bit of history 2/3

- In the early 2000s, many researchers suggested new applications of negative materials.
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- L. Chesnel, Étude de quelques problèmes de transmission avec changement de signe. Application aux métamatériaux, Ph.D. Thesis, Ecole Polytechnique X, 2012.

- For polygonal interfaces, the critical interval has been characterized by means of black hole waves.
- A new functional framework has introduced for the case of critical contrasts.
- The introduction of T-conforming meshes.
- If the scalar problem is well-posed then the Maxwell's one is well-posed.

Introduction: a bit of history

- C. Carvalho, Étude mathématique et numérique de structures plasmoniques avec coins,Ph.D. Thesis, Ecole Polytechnique X, 2015

- The construction of T-conforming meshes for interfaces with corners.
- The use of PML for the approximation of the solution for critical contrasts.

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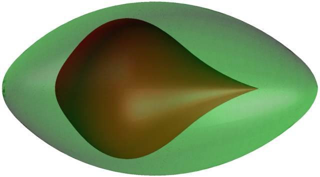
- C. Carvalho, Étude mathématique et numérique de structures plasmoniques avec coins,Ph.D. Thesis, Ecole Polytechnique X, 2015
- Mahran Rihani, Maxwell's equations in presence of metamaterials, Institut polytechnique de Paris,Ph.D. Thesis, 2022.

- The study of the scalar problem and the Maxwell's one for the case of interfaces with conical tips.
- Optimal-control based numerical method for the scalar problem.



Introduction: a bit of history 3/3

- C. Carvalho, Étude mathématique et numérique de structures plasmoniques avec coins,Ph.D. Thesis, Ecole Polytechnique X, 2015
- Mahran Rihani, Maxwell's equations in presence of metamaterials, Institut polytechnique de Paris,Ph.D. Thesis, 2022.
- In parallel with these developments, significant progress has been made in the analysis of the essential spectrum of the Neumann-Poincaré operator for the case of non-smooth interfaces (eg. Perfekct et al).

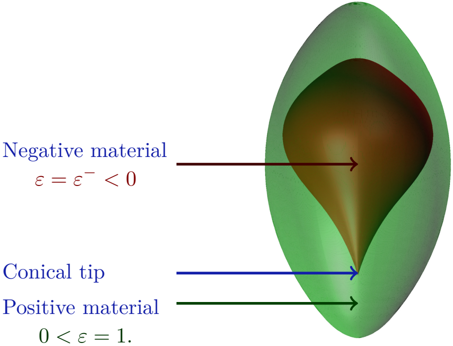
$$\sigma_{\text{ess}}(\text{NP}, \text{H}^{-1/2}(\Sigma)) = F(I_\Sigma), \text{F is known explicitly}.$$

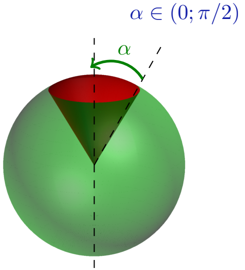
- Combining the volumic approach (ie. T-coercivity) and the integral one leads to interesting results:
  1. Compactness of NP for $\mathcal{C}^1$ interfaces.
  2. $\sigma_{\text{ess}}(\text{NP}, \text{H}^{-1/2}(\Sigma)) = -\sigma_{\text{ess}}(\text{NP}, \text{H}^{-1/2}(\Sigma))$ in 2D (Costabel's talk on P. Joly's 60th birthday).
  3. Characterization of I_Σ for conical tips, ...

Main goal of the talk

- Scattering of electromagnetic waves, in time harmonic regime, by a positive-negative medium: $\Omega = \Omega^+ \cup \Omega^- \cup \Sigma \subset \mathbb{R}^3$.

Find $\boldsymbol{E}$ such that $\left| \begin{array}{ll} \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E} - \omega^2\varepsilon\boldsymbol{E} = i\omega\boldsymbol{J} & \text{in } \Omega \\ \boldsymbol{E}\times\nu = 0 & \text{on } \partial\Omega. \end{array} \right.$





A zoom near the origin.

We want to study the behaviour of the electric field near the conical tip.

State of the art

2014: A.-S. Bonnet-Ben Dhia, P. Ciarlet, L. Chesnel **T-coercivity for the Maxwell problem with sign-changing coefficients**.

- For a general setting, if $\varepsilon^- \notin I_\Sigma$(the critical interval) then $\boldsymbol{E}$ has a classical behaviour: $\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{E} \in \mathbf{L}^2$ (finite energy).

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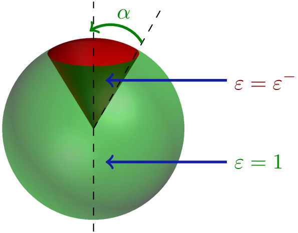
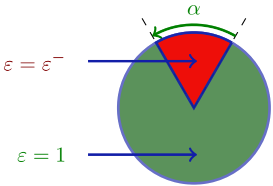


- $I_\Sigma = I_\alpha^{3D}$.
- How to determine explicitly I_α^{3D}?

What happens when $\varepsilon^- \in I_\alpha^{3D}$?

2012: Bonnet-Ben Dhia, Chesnel, Claeys, **Radiation condition for a non-smooth interface between a dielectric and a metamaterial (the 2D scalar case).**



$$-\mathrm{div}\,(\varepsilon \nabla u) = f \in \mathrm{L}^2.$$

- When $\varepsilon^- \notin I_\alpha^{2D} = [(\alpha - 2\pi)/\alpha; \alpha/(\alpha - 2\pi)]$ then u has a classical behaviour: $u, \nabla u \in \mathrm{L}^2$.
  - $I_{\pi/2}^{2D} = [-3; -1/3]$, $I_{\alpha \to 0}^{2D} = (-\infty; 0)$.

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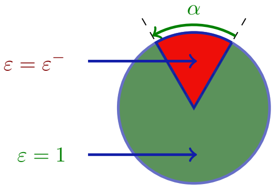

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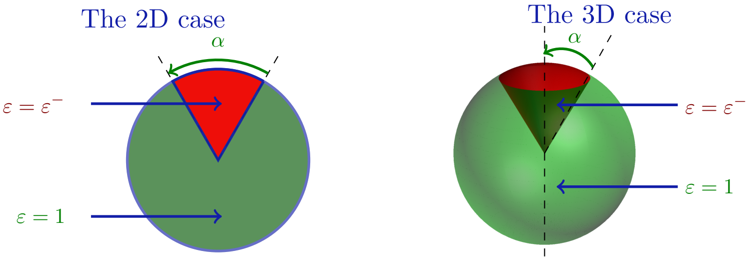
- When $\varepsilon^- \notin I_\alpha^{2D} = [(\alpha - 2\pi)/\alpha; \alpha/(\alpha - 2\pi)]$ then u has a classical behaviour: $u, \nabla u \in \mathrm{L}^2$.
  - $I_{\pi/2}^{2D} = [-3; -1/3]$, $I_{\alpha \to 0}^{2D} = (-\infty; 0)$.
- If $\varepsilon^- \in I_\alpha^{2D} \backslash \{-1\}$, the physical solution is not of finite energy $\nabla u \notin \mathbf{L}^2$. $\Longrightarrow$ `A new functional framework for the scalar problem.`

What happens if we consider the 2D Maxwell's equations with $\varepsilon^- \in I_\alpha^{2D}$?

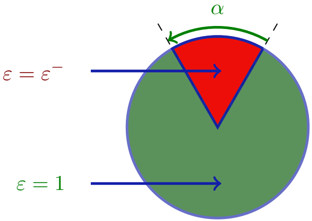
Main result: the 2D/3D Maxwell's equations



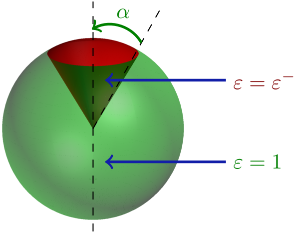
If $\varepsilon^- \in I_\alpha^{2D} \setminus \{-1\}$ in 2D or if $\varepsilon^- \in I_\alpha^{3D} \setminus \{-1\}$ in 3D then
$\boldsymbol{E} = \nabla \boldsymbol{S} + \tilde{\boldsymbol{E}}$ with $\nabla \boldsymbol{S} \notin \mathrm{L}^2$ and $\tilde{\boldsymbol{E}} \in \mathrm{L}^2$.

Main result: the 2D/3D Maxwell's equations

The 2D case



The 3D case



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$\boldsymbol{E} = \nabla \boldsymbol{S} + \tilde{\boldsymbol{E}}$ with $\nabla \boldsymbol{S} \notin \mathrm{L}^2$ and $\tilde{\boldsymbol{E}} \in \mathrm{L}^2$.

- The singular part $\nabla \boldsymbol{S}$ satisfies $\operatorname{div}(\varepsilon \nabla \boldsymbol{S}) = 0$.
- $\boldsymbol{S}$ belongs to a finite dimensional space called the space of propagating singularities.

- Theoretical issues: absence of coercivity, working with non L^2 fields, non-self-adjoint problems, ...
- Numerical issues: Classical methods are not convergent and unstable.

Outline of the talk

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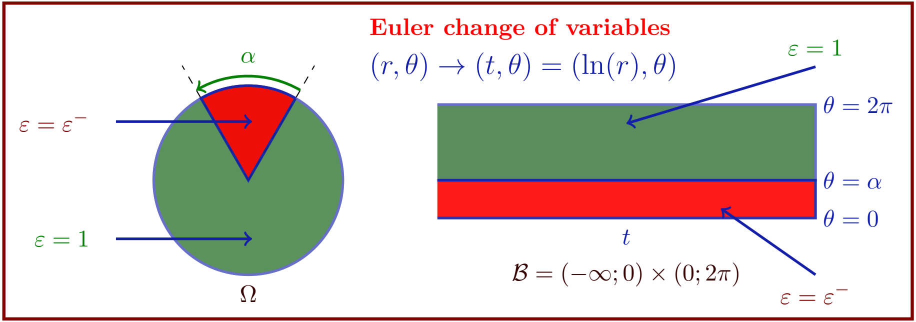
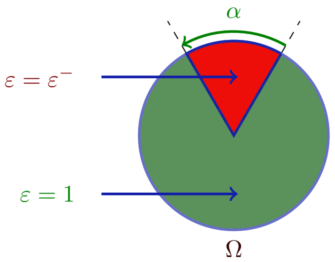
The propagating singularities (the 2D case) 1/4



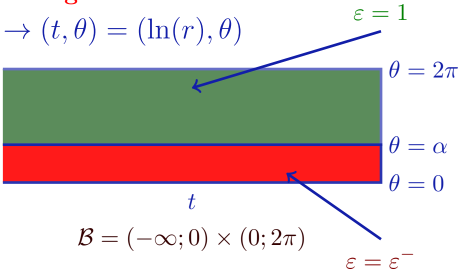
- $-\mathrm{div}\,(\varepsilon \nabla u) = f$ in Ω becomes $-\mathrm{div}_{\,t,\theta}(\varepsilon \nabla u) = \mathrm{e}^{2t} f$ in $\mathcal{B}$ (a waveguide problem!).

The propagating singularities (the 2D case) 1/4



Euler change of variables

$(r, \theta) \to (t, \theta) = (\ln(r), \theta)$



- $-\mathrm{div}\,(\varepsilon\nabla u) = f$ in Ω becomes $-\mathrm{div}_{\,t,\theta}(\varepsilon\nabla u) = \mathrm{e}^{2t} f$ in $\mathcal{B}$ (a waveguide problem!).
- Formally, the physical solution must have the form

$$u(t,\theta) = \sum_{\Re e(\lambda)\geq 0} \mathrm{e}^{\lambda t}\varphi_\lambda(\theta) + \cdots \Longrightarrow u(r,\theta) = \sum_{\Re e(\lambda)\geq 0} r^\lambda \varphi_\lambda(\theta) + \ldots$$

- The functions $\mathrm{e}^{\lambda t}\varphi_\lambda(\theta)$ (or $r^\lambda\varphi_\lambda(\theta)$) are called modes of the problem.

The propagating singularities (the 2D case) 2/4

- The spectrum of the transverse operator

$$\text{Find } (\varphi_\lambda, \lambda) \text{ such that } \int_0^{2\pi} \varepsilon \varphi_\lambda' \varphi' \, d\theta = \lambda^2 \int_0^{2\pi} \varepsilon \varphi_\lambda \varphi \, d\theta \text{ for all } \varphi.$$

 Non-self-adjoint problem because of the sign change in ε.

The propagating singularities (the 2D case) 2/4

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Non-self-adjoint problem because of the sign change in ε.

- Negative eigenvalue $\Longrightarrow$ Propagating modes: $e^{i\eta t}\varphi(\theta)$ in $\mathcal{B}$ with $\eta \in \mathbb{R}^*$.

- $e^{i\eta t}\varphi(\theta)$ in $\mathcal{B} \longleftrightarrow r^{i\eta}\varphi(\theta)$ in Ω.
- $\nabla r^{i\eta}\varphi(\theta) \notin L^2$.


When can we have propagating modes?

The propagating singularities (the 2D case) 3/4

- Dispersion relation: propagating modes exist if and only if

$$(\varepsilon^- \tanh(\eta\gamma) + \tanh(\eta))(\varepsilon^- \tanh(\eta) + \tanh(\eta\gamma)) = 0; \quad \gamma = \frac{2\pi - \alpha}{\alpha}.$$

- "Simplified" dispersion relation: propagating modes exist if and only if

$$\varepsilon^- \in I_\alpha^{2D} = [\frac{\alpha - 2\pi}{\alpha}, \frac{\alpha}{\alpha - 2\pi}]\backslash\{-1\}.$$

The propagating singularities (the 2D case) 3/4

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$$\varepsilon^- \in I_\alpha^{2D} = [\frac{\alpha - 2\pi}{\alpha}, \frac{\alpha}{\alpha - 2\pi}]\backslash\{-1\}.$$

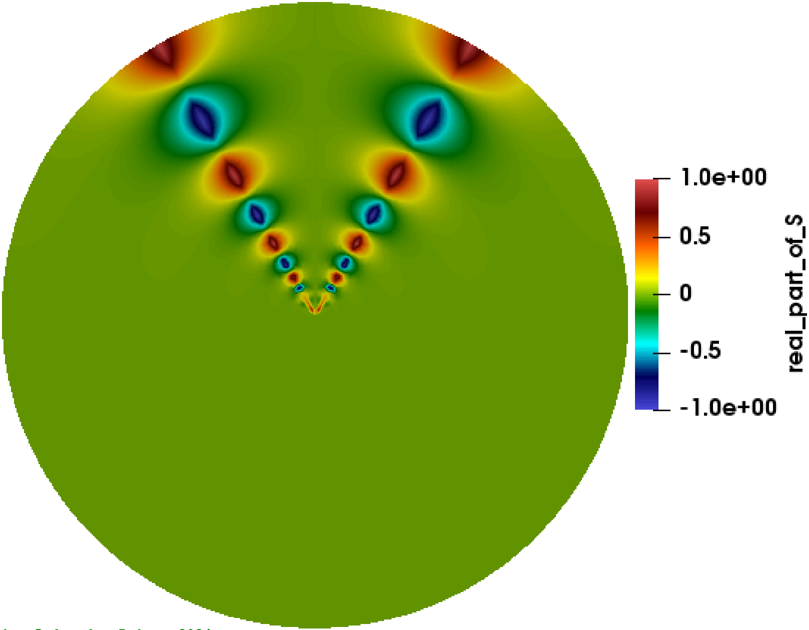
- For all $\varepsilon^- \in I_\alpha^{2D}$ just one "propagating singularities" exist:

$$\mathfrak{s}^+(r,\theta) = r^{i\eta}\varphi(\theta) \quad \eta \in \mathbb{R}^*.$$

The physical solution of $-\text{div}\,(\varepsilon\nabla u) = f$ in Ω decomposes as

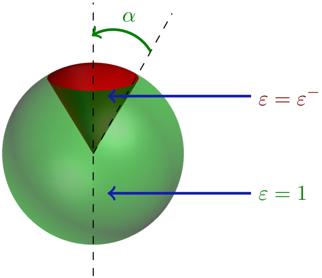
$$u = c\,\mathfrak{s}^+ + \tilde{u} \Longrightarrow \boldsymbol{E} = c\nabla\mathfrak{s}^+ + \tilde{\boldsymbol{E}}.$$

The propagating singularities (the 2D case) 4/4



Outline of the talk

The propagating singularities (the 3D case) 1/3



Euler change of variables:

- $(r, \theta, \varphi) \to (t, \theta, \varphi) = (\ln(r), \theta, \varphi)$.
- $\Omega \to \mathcal{B} = (-\infty; 0) \times \mathbb{S}^2$.

$-\mathrm{div}\,(\varepsilon \nabla u) = f$ in Ω becomes $L(u) = \hat{f}$ in $\mathcal{B}$.

- $L(u) = e^{t/2}(\varepsilon \partial_t^2 + \varepsilon \partial_t - \delta_\varepsilon)(u)$ and $\hat{f} = e^{3t/2} f$ ($\hat{f} \in L^2(\mathcal{B})$ if $f \in L^2(\Omega)$)

$$\delta_\varepsilon := \frac{1}{\sin(\theta)} \partial_\theta (\varepsilon \sin(\theta) \partial_\theta) + \frac{1}{\sin(\theta)^2} \partial_\varphi \varepsilon \partial_\varphi.$$

- The physical solution admits the decomposition

$$u(t, \theta) = \sum_{\Re e(\lambda) \geq -1/2} e^{\lambda t} \Phi_\lambda(\theta) + \cdots \Longrightarrow u(r, \theta) = \sum_{\Re e(\lambda) \geq -1/2} r^\lambda \Phi_\lambda(\theta) + \ldots$$

The propagating singularities (the 3D case) 2/3

- Propagating modes are of the form $\mathrm{e}^{(-1/2+i\eta)t}\Phi(\theta,\varphi)$.
- Propagating modes exist if and only if $\varepsilon^- \in I_\alpha^{3D} = (-1; -a_\alpha)$

$$a_\alpha = \frac{{}_2\mathrm{F}_1(1/2, 1/2, 1, \cos^2(\alpha/2))\, {}_2\mathrm{F}_1(3/2, 3/2, 2, \sin^2(\alpha/2))}{{}_2\mathrm{F}_1(1/2, 1/2, 1, \sin^2(\alpha/2))\, {}_2\mathrm{F}_1(3/2, 3/2, 2, \cos^2(\alpha/2))}$$

${}_2\mathrm{F}_1$ is the Gaussian hypergeometric function.

- For all $\varepsilon^- \in (-1; -a_\alpha)$ there exists $1 \leq N(\varepsilon^-)$ outgoing propagating modes:

$$1 \leq N(\varepsilon^-) \leq +\infty.$$

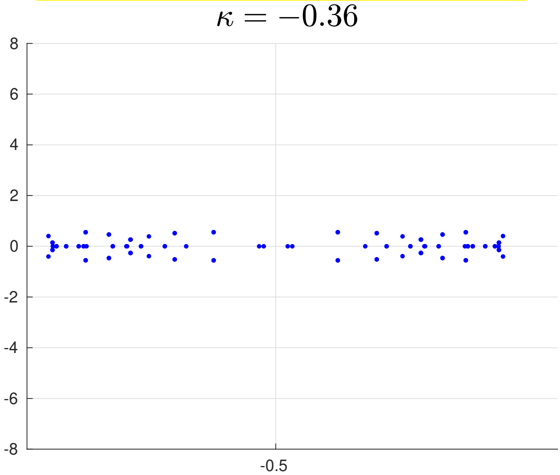
The physical solution of $-\mathrm{div}\,(\varepsilon \nabla u) = f$ decomposes as

$$u(r,\theta) = \sum_{n=1}^{N(\varepsilon^-)} c_n \mathfrak{s}_n + \tilde{u} \Longrightarrow \boldsymbol{E} = \sum_{n=1}^{N(\varepsilon^-)} c_n \nabla \mathfrak{s}_n + \tilde{\boldsymbol{E}}$$

in which $\mathfrak{s}_n(r,\theta,\varphi) = r^{-1/2+i\eta_n}\Phi(\theta,\varphi)$.

The propagating singularities (the 3D case) 3/3

When $\alpha = \pi/3$, $(-1; -a_\alpha) = (-1; -0.386)$



Outside the critical interval.

The propagating singularities (the 3D case) 3/3

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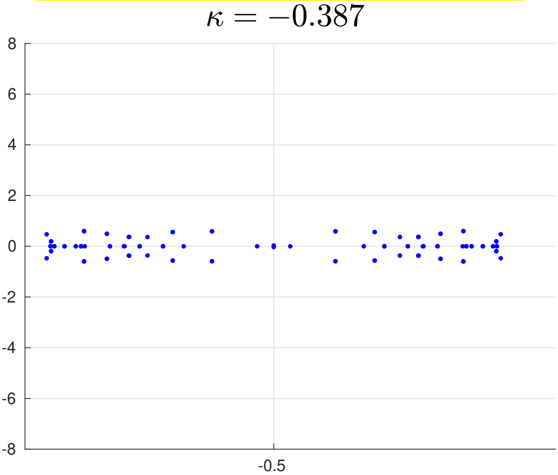

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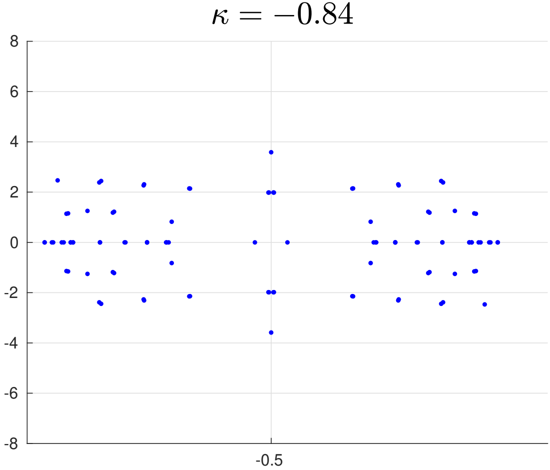

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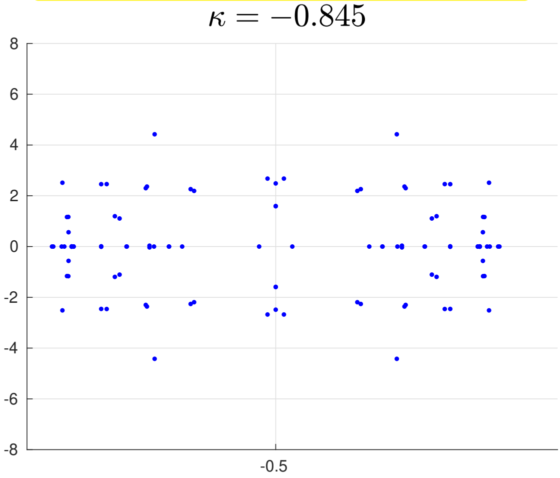

Inside the critical interval.

The propagating singularities (the 3D case) 3/3

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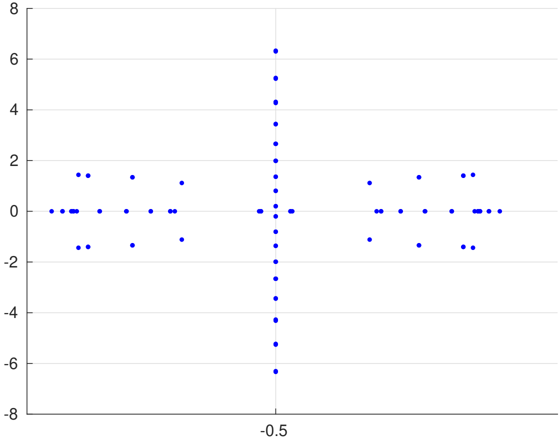
$\kappa = -0.99$



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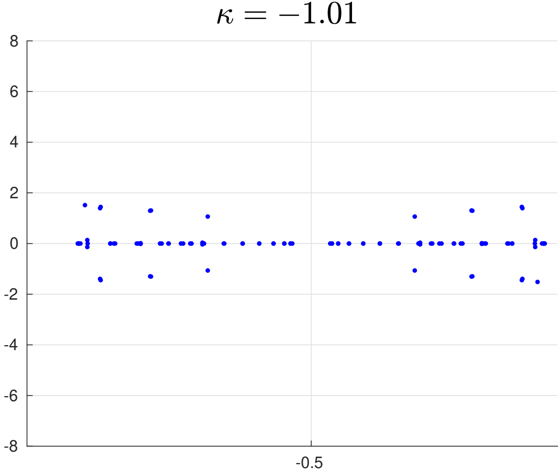
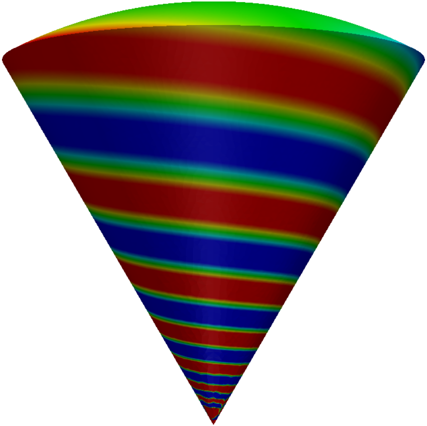
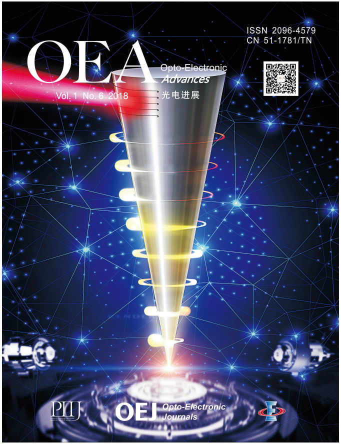


Outside the critical interval.

An example of propagating singularity





- The propagating singularities can be interpreted as surface waves that are guided by the surface Σ. The tip plays the role of infinity.
- Some of the propagating singularities are outgoing, some of them are incoming and it is possible to have standing ones.

The 2D case versus the 3D case

The 2D case:

- $I_\alpha^{2D} = (-\gamma, -1/\gamma)$ is symmetric w.r.t -1.
- The number of propagating singularities is independent of ε^-
- Just one propagating singularities.
- $\mathfrak{s}(r,\theta) = r^{i\eta}\varphi(\theta), \eta \in \mathbb{R}$.
- When $\alpha \to 0$ $I_\alpha^{3D} \to (-\infty; 0)$.

The 3D case:

- $I_\alpha^{3D} = (-1, -a_\alpha)$ is not symmetric w.r.t -1.
- The number of propagating singularities depends on ε^-
- $N(\varepsilon^-)$ propagating singularities: $1 \leq N(\varepsilon^-) \leq +\infty$.
- $\mathfrak{s}(r,\theta) = r^{-1/2+i\eta}\varphi(\theta), \eta \in \mathbb{R}^*$.
- When $\alpha \to 0$, $I_\alpha^{3D} \to (-1; 0)$

Outline of the talk

Concluding remarks

What is done ?

- We studied the scalar problem between a positive and a negative material that are separated by interface with a smooth conical tip.
- We explained how to construct adapted physical framework when propagating singularities exist.
- We also considered the time harmonic Maxwell's equations is this configuration.

Future works

- Extension of our results to other singular geometries (3D interfaces with edges).
- Numerical approximation of problem: how to deal with propagating singularities for ?

Thank you for your attention!!!