Numerical modelling of open elastic waveguides for their non-destructive evaluation

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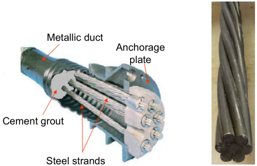
Generality

Guided wave applications:

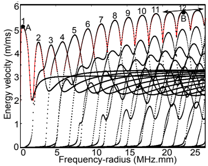
- dynamic analysis of elongated structures: **N**on **D**estructive **E**valuation (ultrasonic), noise and vibration reduction...
- **our flagship application:** NDE of bridge cables
- potentialities:
  - propagation over long distances
  - sensibility to small damages



Cable anchorage and 7-wire strand

Complexity of guided waves:

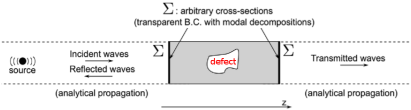
- dispersive and multimodal propagation
- dispersion curves required
- modeling tools mandatory



Energy velocity vs. frequency in a cylindrical bar

Three modeling issues:

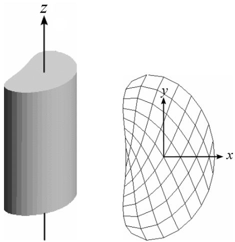
- propagation of waves
- generation by a source
- scattering by a local inhomogeneity



Tool #1: propagation

SAFE method: (**S**emi-**A**nalytical **F**inite **E**lement)

1. variational formulation for 3D elastodynamics:
   $\int_\Omega \delta\boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{C}\boldsymbol{\epsilon}\mathrm{d}V - \omega^2\int_\Omega \rho\delta\mathbf{u}^{\mathrm{T}}\mathbf{u}\mathrm{d}V = \int_\Omega \rho\delta\mathbf{u}^{\mathrm{T}}\mathbf{f}\mathrm{d}V$
   où $\boldsymbol{\epsilon} = (\mathbf{L}_{xy} + \mathbf{L}_z\partial/\partial z)\mathbf{u}$
2. Fourier transform along z: $\hat{\mathbf{u}}(k) = \int_{-\infty}^{+\infty}\mathbf{u}(z)\mathrm{e}^{-\mathrm{i}kz}$
   $\rightarrow$ continuous symmetry incorporated
3. FE discretization of cross-section (x, y):
   $\{\mathbf{K}_1 - \omega^2\mathbf{M} + \mathrm{i}k(\mathbf{K}_2 - \mathbf{K}_2^{\mathrm{T}}) + k^2\mathbf{K}_3\}\hat{\mathbf{U}}(k;\omega) = \hat{\mathbf{F}}(k;\omega)$
   $\rightarrow$ 2D problem, iteration over frequency ω
4. free response ($\hat{\mathbf{F}} = \mathbf{0}$)
   $\rightarrow$ quadratic eigenvalue problem
   $\rightarrow$ solution = wave modes $\{k_m, \mathbf{U}_m\}$ $\rightarrow$ linearized form:
   $$\left(\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -(\mathbf{K}_1 - \omega^2\mathbf{M}) & -\mathrm{j}(\mathbf{K}_2 - \mathbf{K}_2^{\mathrm{T}}) \end{bmatrix} - k\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_3 \end{bmatrix}\right)\begin{bmatrix} \hat{\mathbf{U}} \\ k\hat{\mathbf{U}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$



From a 3D waveguide to its 2D SAFE mesh

Remark: matrices can be complex (viscoelasticity, PML...)

Tool #2: generation

$\mathbf{K}_1$, $\mathbf{K}_3$ et $\mathbf{M}$ are symmetric, $(\mathbf{K}_2 - \mathbf{K}_2^T)$ is skew-symmetric

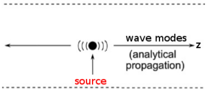
Biorthogonality

- if k_m is an eigenvalue, then $-k_m$ also
  $\Rightarrow$ pairs of eigenmodes traveling in opposite direction $\{k_m, \mathbf{U}_m\}$ and $\{-k_m, \mathbf{U}_{-m}\}$
- the biorthogonality relationship can be written as:
  $\mathrm{i}\frac{\omega}{4}\left(\mathbf{U}_m^T\mathbf{F}_{-n} - \mathbf{U}_{-n}^T\mathbf{F}_m\right) = Q_{m,-m}\delta_{mn}$

with $\mathbf{F}_m = (\mathbf{K}_2^T + \mathrm{i}k_m\mathbf{K}_3)\mathbf{U}_m$ (eigenforce vector)

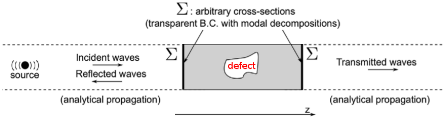
Forced response:

1. modal expansion: $\hat{\mathbf{U}}(k;\omega) = \sum_{m=-M}^{+M} \hat{\beta}_m(k;\omega)\mathbf{U}_m(\omega)$
2. **biorthogonality**+residue theorem+time inverse FT:
   $$\mathbf{U}(z;t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{m=1}^{M}\mathbf{E}_m(\omega)\hat{\mathbf{F}}(k_m;\omega)e^{\mathrm{i}k_m(\omega)z}\,\mathrm{e}^{-\mathrm{i}\omega t}d\omega$$
   with $\mathbf{E}_m = \frac{\mathrm{i}\omega}{4Q_{m,-m}}\mathbf{U}_m\mathbf{U}_{-m}^T$ (**excitability** of mth mode)



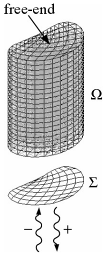
Tool #3: scattering

Hybrid FE-SAFE method: a small FE box with transparent boundary conditions



$$\mathbf{u}|_\Sigma = \sum_{n=-\infty}^{+\infty} \alpha_n \mathbf{u}_n, \quad \mathbf{t}|_\Sigma = \sum_{n=-\infty}^{+\infty} \alpha_n \mathbf{t}_n$$

1. FE model of the small box including the inhomogeneity: $\delta \mathbf{U}^T(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{U} = \delta \mathbf{U}^T \mathbf{F}$
2. Partitioning of dofs into cross-section Σ and internal region I :
$$\mathbf{U} = \begin{Bmatrix} \mathbf{U}_\Sigma \\ \mathbf{U}_I \end{Bmatrix}, \quad \mathbf{F} = \begin{Bmatrix} \mathbf{F}_\Sigma \\ \mathbf{F}_I \end{Bmatrix} \quad \text{avec } I = \Omega \setminus \Sigma$$



FE-SAFE mesh

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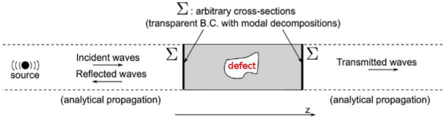

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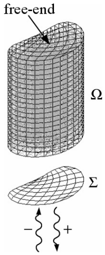
   $$\mathbf{U} = \begin{Bmatrix} \mathbf{U}_\Sigma \\ \mathbf{U}_I \end{Bmatrix}, \quad \mathbf{F} = \begin{Bmatrix} \mathbf{F}_\Sigma \\ \mathbf{F}_I \end{Bmatrix} \quad \text{avec } I = \Omega \setminus \Sigma$$
3. **Modal expansion of both $\mathbf{U}_\Sigma$ and $\mathbf{F}_\Sigma$:**

$$\mathbf{U}_\Sigma = \sum_{n=1}^{N} \alpha_n^- \mathbf{U}_n^- + \sum_{n=1}^{N} \alpha_n^+ \mathbf{U}_n^+, \quad \mathbf{F}_\Sigma = \sum_{n=1}^{N} \alpha_n^- \mathbf{F}_n^- + \sum_{n=1}^{N} \alpha_n^+ \mathbf{F}_n^+$$

$\alpha_n^\pm$: outgoing/ingoing modal coeff. (**unknown**/**prescribed**)
N: number of modes kept (after **truncation**)

4. Linear system of the following form:
   $$\mathbf{A}(\omega)\mathbf{x}(\omega) = \mathbf{B}(\omega)\mathbf{y}(\omega), \text{ with } \mathbf{x} = \begin{Bmatrix} \boldsymbol{\alpha}^+ \\ \mathbf{U}_I \end{Bmatrix} \text{ and } \mathbf{y} = \begin{Bmatrix} \boldsymbol{\alpha}^- \\ \mathbf{F}_I \end{Bmatrix}$$



FE-SAFE mesh

Remarks

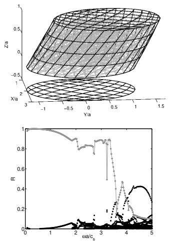
Remarks on biorthogonality:

- 'general', in particular applicable to:
  - non-propagative modes
  - lossy waveguides (including **PML**)
  - full anisotropy (including **curvature**)
- nothing but the discrete version of Auld's real relationship[1]
- degenerates to more specific but well-known relations:
  - Auld's complex relation (applicable to real modes only)
  - Fraser's, JASA, 1976 (orthotropic materials only), foundation of **X-Y** formalism
  - Herrera's, BSSA, 1964 (surface waves in 1D stratified media)

[1] Auld, Acoustic Fields and Waves in Solids, 1990

Remarks on hybrid FE-SAFE approach:

- no specific hyp. (anisotropy, loss ok)
- consequence of biorthogonality: $\mathbf{A}(\omega)$ is symmetric
- **consistency:**
  - cross-section SAFE mesh: extracted from the FE box
  - explicit expression of traction: the eigenforce $\mathbf{F}_m$
- **error due to mode truncation:**
  - keep the least attenuated mode? a 'natural' criterion: $e^{-|\mathrm{Im}(k_n^{\pm})d|} < \delta$ (d: distance damage-extremity)
  - but not always relevant for open waveguides...

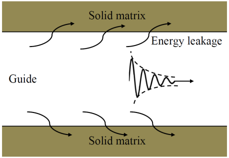


Reflection coeff. by a helix free edge ϕ=15°

Buried waveguides (collaboration/POEMS: PhD theses K.L. Nguyen 2011-14 + M. Gallezot 2015-18)

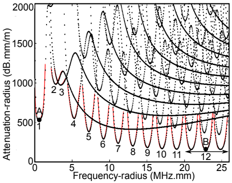
Waveguides coupled to an infinite surrounding medium:

- unbounded in the transverse direction
- terminology: **open waveguides** (as opposed to *closed waveguides*, in vacuum)
- **NDE of buried waveguides: minimize leakage**

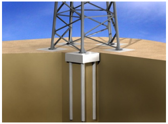
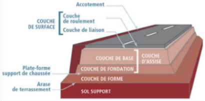


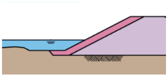
A more complex physics, with:

- trapped modes, perfectly guided... a discrete set often empty (depending on materials)
- radiation modes... which form a continuous set
- **leaky modes**, axially decreasing due to radiation loss... but growing to ∞ in the transverse direction





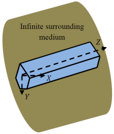


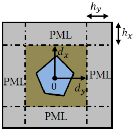
Exemples of open waguides in civil engineering (fully or partially buried)

Free response: extension of tool #1

Method selection : SAFE+PML (perfectly matched layers)

1. **PML** = analytical continuation of transverse coordinates (x, y):
   $\tilde{x} = \int_0^x \gamma_x(s)ds$ avec $\begin{cases} \gamma_x = 1 & \text{if } |x| \leq d_x \\ \mathrm{Im}\,\gamma_x > 0 & \text{si } |x| > d_x \end{cases}$ (same for $\tilde{y}$)
2. change from complex to real coordinates:
   $\tilde{x} \mapsto x : \quad \frac{\partial}{\partial \tilde{x}} = \frac{1}{\gamma_x}\frac{\partial}{\partial x}, \quad d\tilde{x} = \gamma_x dx$ (same for $\tilde{y}$)
3. PML truncation to a finite thickness (closed problem)
4. **SAFE+PML** method leads to:
   $\{\mathbf{K}_1 - \omega^2\mathbf{M} + ik(\mathbf{K}_2 - \mathbf{K}_2^T) + k^2\mathbf{K}_3\}\hat{\mathbf{U}}(k;\omega) = \hat{\mathbf{F}}(k;\omega)$
   - complex matrices due to γ_x, γ_y
   - problem is 'definitively' not self-adjoint





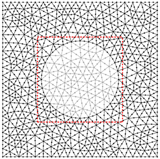
A rather easy implementation

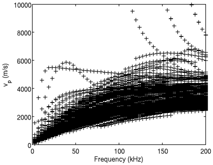
but 3 user-defined parameters:
absorbing function $\gamma_x(x)$, interface distance d_x, thickness h_x (same for y)

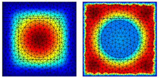
Free response: example

Example: a steel cylindrical bar buried into a soft medium (concrete)

Dispersion curves:







Leaky mode vs. PML mode

SAFE-PML mesh, dispersion curves before filtering and after

- many 'PML modes', non intrinsic to the physics...
- an energy-based modal filtering: $E_{PML}/E_{TOT} >$ threshold

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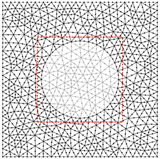

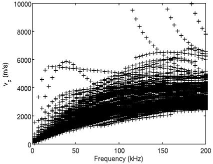

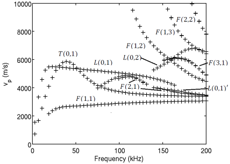

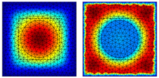

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- many 'PML modes', non intrinsic to the physics...
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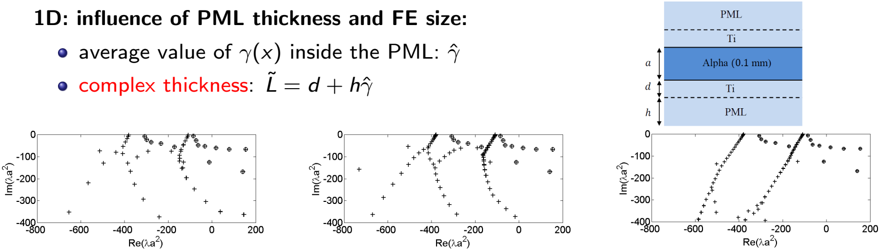
Free response: a closer look at spectrum

1D: influence of PML thickness and FE size:

- average value of $\gamma(x)$ inside the PML: $\hat{\gamma}$
- complex thickness: $\tilde{L} = d + h\hat{\gamma}$



thickness: h, FE size: ℓ_e — $h\times 2$, ℓ_e — $h\times 2$, $\ell_e/2$

Spectrum $\lambda = -k^2$, ○: analytical leaky modes

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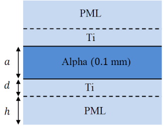
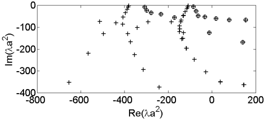
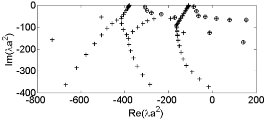
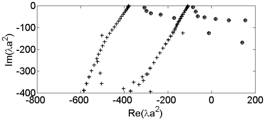




thickness: h, FE size: ℓ_e



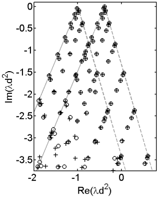
$h \times 2$, ℓ_e



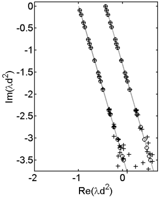
$h \times 2$, $\ell_e/2$

Spectrum $\lambda = -k^2$, ○: analytical leaky modes

2D: the homogeneous test case with mixed bc → analytical solution available



$\arg(\tilde{L}_x) \neq \arg(\tilde{L}_y)$



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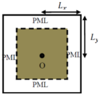


- PML modes lay inside 2 sectors, 'usually' degenerating to 2 half-lines as in 1D
- rotation angles of half-lines $\simeq -2\arg(\tilde{L}_x)$, $-2\arg(\tilde{L}_y)$

History break

PhD Khac Long Nguyen (2011-2015)

Papers:

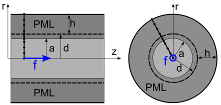
- Nguyen, K. L. and Treyssède, F. and Hazard, C., Numerical modeling of three-dimensional open elastic waveguides combining semi-analytical finite element and perfectly matched layer methods, Journal of Sound and Vibration 344 (2015), 158-178
- Treyssède, F. and Nguyen, K. L. and Bonnet-BenDhia, A. S. and Hazard, C., Finite element computation of trapped and leaky elastic waves in open stratified waveguides, Wave Motion 51 (2014), 1093-1107

Conferences:

- Nguyen, K. L. and Treyssède, F. and Bonnet-BenDhia, A.-S. and Hazard, C., Finite element computation of leaky modes in straight and helical elastic waveguides, 8th GDR US Conference, Gregynog (Wales), 2014
- Nguyen, K. L. and Treyssède, F. and Bonnet-BenDhia, A.-S. and Hazard, C., Modélisation numérique des guides d'onde ouverts : cas des structures élastiques courbes, 12ème CFA, Poitiers, 2014
- Nguyen, K. L. and Treyssède, F. and Bonnet-BenDhia, A.-S. and Hazard, C., Computation of leaky modes in three-dimensional open elastic waveguides, Waves, Tunis, 2013
- Nguyen, K. L. and Treyssède, F. and Bonnet-BenDhia, A.-S. and Hazard, C., Computation of dispersion curves in elastic waveguides of arbitrary cross-section embedded in infinite solid media, 13th International Symposium on Nondestructive Characterization of Materials, Le Mans, 2013
- Treyssède, F. and Nguyen, K. L. and Bonnet-BenDhia, A.-S. and Hazard, C., On the use of a SAFE-PML technique for modeling two-dimensional open elastic waveguides, Acoustics 2012, Nantes
- Treyssède, F. and Nguyen, K. L. and Bonnet-BenDhia, A.-S. and Hazard, C., Finite element computation of elastic propagation modes in open stratified waveguides, 7th GDR US Conference, Oléron, 2012

Forced response (tool #2): a numerical experiment

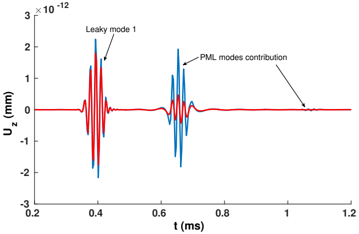
Example: steel cylindrical bar buried into a soft medium excited by a point force



Axisymmetric SAFE-PML model, PML parameters: $\hat{\gamma} = 4 + 4i$, $h = 4a$, $d = a$

Free response: 1 leaky mode, 0 trapped, 50 PML modes (low-frequency regime)

Forced response:



Radial displacement vs. time at $z = 175a$, elastic (blue) and viscoelastic (red) material properties

What is the physical meaning, if any, of the contribution of PML modes?

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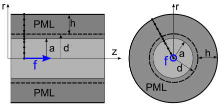

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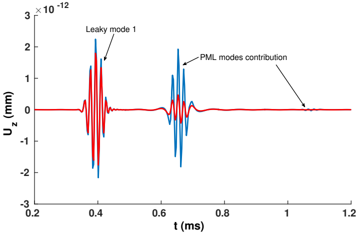
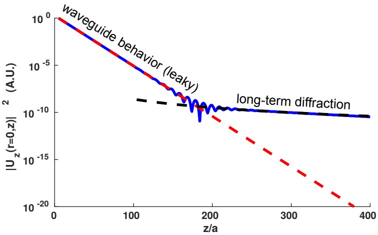


Radial displacement vs. time at $z = 175a$, elastic (blue) and viscoelastic (red) material properties

What is the physical meaning, if any, of the contribution of PML modes?



energy vs. distance
(blue: with PML modes, red: without)

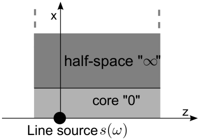
contribution of PML modes = wavefield part with geometric decay e^{ikr}/r^{α}

Physics of open waveguides: 2D scalar toy model

Solution in the half-space:

$$u_\infty(x,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{a(k)}{b(k)} e^{i\alpha_\infty x} e^{ikz}\, dk$$

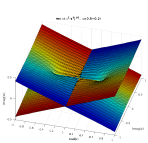
with $\alpha_\infty = \pm\sqrt{\omega^2/c_\infty^2 - k^2}$: multi-valued



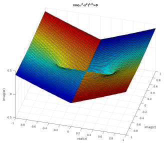
Residue theorem for $z > 0$:

$$u(x,z) = \underbrace{\sum \textbf{trapped modes}^a}_{\text{poles of proper sheet}} + \underbrace{\int_{\Gamma^+} \textbf{radiation modes}}_{\text{branch cut contribution}}$$

[a] Trapped modes do not exist if $c_0 > c_\infty$, i.e. for our usual configuration...



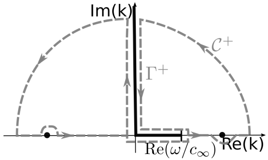
Riemann surface for α_∞



Proper sheet $\text{Im}(\alpha_\infty) > 0$

Riemann sheet to get $\int_{\mathcal{C}+} \to 0$: $\text{Im}(\alpha_\infty) > 0$

$\text{Im}(\alpha_\infty) = 0$: discontinuity (branch cut)



Integration path, Γ^+: branch cut

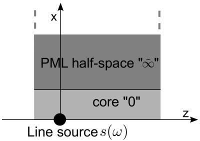
•: poles of proper sheet (trapped)

Physics of open waveguides: 2D scalar toy model

With an infinite PML and cst $\gamma(x) = \gamma$:

$$u_\infty(x,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{a(k)}{b(k)} e^{i\gamma\alpha_\infty x} e^{ikz} dk$$

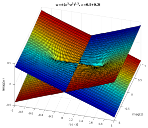
with $\alpha_\infty = \pm\sqrt{\omega^2/c_\infty^2 - k^2}$: multi-valued



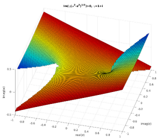
Residue theorem for $z > 0$:

$$u(x,z) = \sum \text{modes piégés}^a + \underbrace{\sum \textbf{revealed leaky modes}^b}_{\text{poles such that } \mathrm{Im}(\alpha_\infty) < 0\,!} + \underbrace{\int_{\tilde{\Gamma}+} \textbf{radiation modes}}_{\text{new branch cut contribution}}$$

[b] Leaky modes are a good approximation of the initial continuum near the core

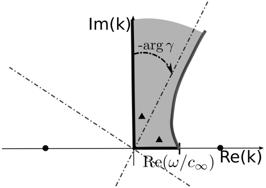


Riemann surface for α_∞



New sheet $\mathrm{Im}(\gamma\alpha_\infty) > 0$

Riemann sheet to get $\int_{\mathcal{C}+} \to 0$: $\mathrm{Im}(\gamma\alpha_\infty) > 0$
$\mathrm{Im}(\gamma\alpha_\infty) = 0$: discontinuity (branch cut)



Branch cut rotation by the PML
▲: poles of the improper sheet (leaky)

Back to elasticity and numerical modeling

Elasticity is more complex than with scalar wavefields:

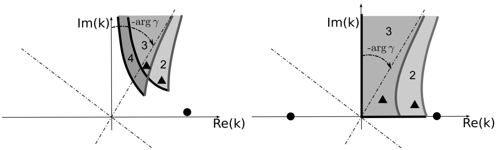
- compressional + shear waves ⇒ 2 transverse wavenumbers, 2 branch cuts
  ⇒ 2 continua of radiation modes, 4 Riemann sheets
- presence of 'backward' leaky modes: in the proper sheet...

Numerical modeling requires PML truncation to a finite thickness:

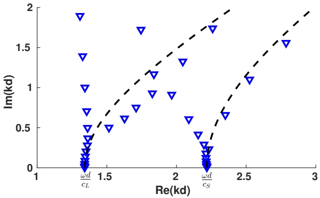
- discretization of continua = discrete set of 'PML modes'
- $u(x, z) = \sum$ trapped $+ \sum$ revealed leaky $+ \sum$ **'PML modes'**(?)

Do PML modes have any physical contribution?

- they are not intrinsic to the physics (depend on user-defined parameters)
- they quickly diverge as their order increases ('spurious modes')



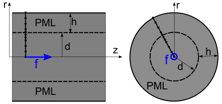
Viscoelastic (left) spectrum vs. elastic (right)
→ only 2 sheets revealed in the elastic case



'Divergence' of discrete set of PML modes
(dashed: PML theoretical branch cut)

Forced response (tool #2): the homogeneous test case

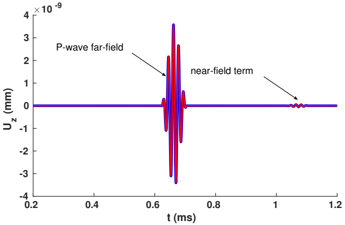
Example: fully homogeneous medium excited by point force pulse (analytical solution available) → no discrete mode, only bulk waves!



Axisymmetric SAFE-PML model of a homogeneous elastic medium
complex thickness $d + \hat{\gamma} h = d + (4 + 4i) \times 4d$

Free response: no trapped, no leaky, 50 PML modes

Forced response:



$u_z(r = 0)$ as a function of time at $z = 175d$
SAFE-PML (red) and analytical solutions (blue)

The exact geometrical decay, e^{ikr}/r^{α}, can be reassembled from the sum of PML modes[a], exponentially decaying ($e^{ik_m z}$)

[a] a proof in scalar waveguides: Olyslager, SIAM J. Appl. Math., 2004

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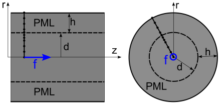

Axisymmetric SAFE-PML model of a homogeneous elastic medium
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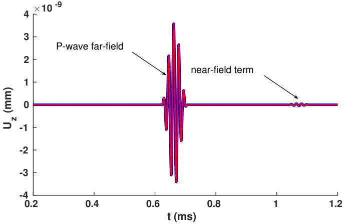

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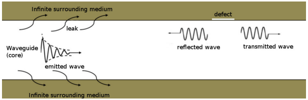
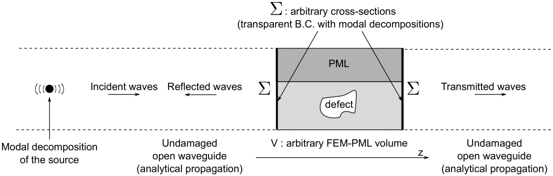
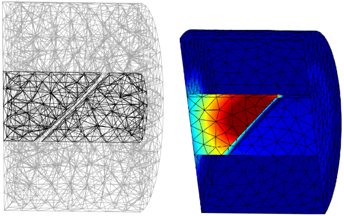
Now, let us go back to our initial experiment...

Scattering (tool #3)





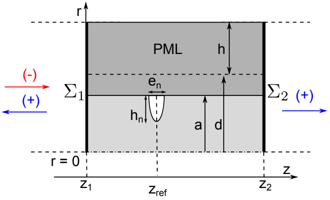
Scattering in buried waveguides: **hybrid FE-SAFE method with PML**

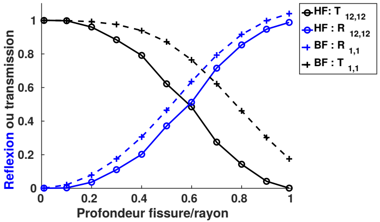


Scattered field by a 3D crack inside an open waveguide (PML-closed)

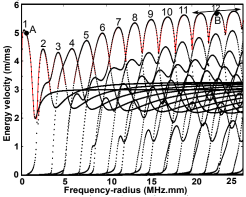
Scattering (tool #3): numerical experiment

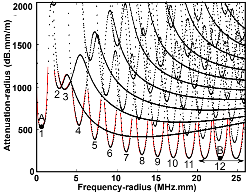
Example: scattering by elliptical crack in a viscoelastic steel cylindrical bar buried into cement grout (softer medium → no trapped modes)
Incident modes: low-frequency L(0,1) vs. high-frequency L(0,12)





Scattering coefficients
(at $z = z_{\text{ref}}$ and normalized by core power)





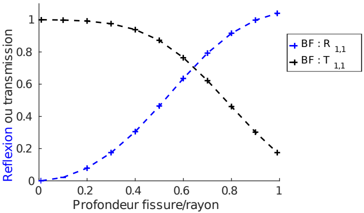
FE box size vs. near field effects and PML modes

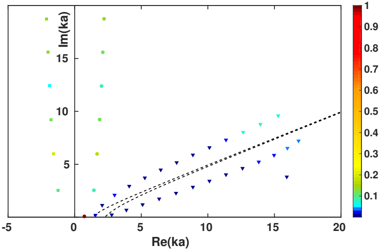
Any contribution of PML modes?

Low-frequency L(0,1) incident:

- $|z_i - z_{\mathrm{ref}}| = 1a$, 13 leaky[a]

[a] including forward and 'backward' modes





Spectrum: forward (circle), backward leaky (square), PML modes (triangle). Color: transmission coeff. ($h_n/a = 0.8$)

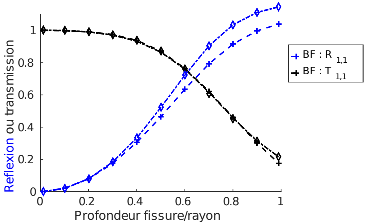
FE box size vs. near field effects and PML modes

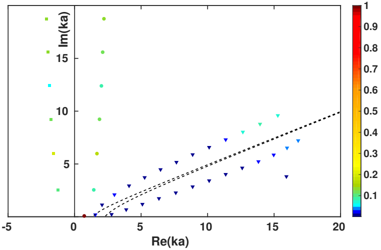
Any contribution of PML modes?

Low-frequency L(0,1) incident:

- $|z_i - z_{\text{ref}}| = 1a$, 13 leaky[a]
- $|z_i - z_{\text{ref}}| = 0.25a$, 13 leaky[a]

[a] including forward and 'backward' modes





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FE box size vs. near field effects and PML modes

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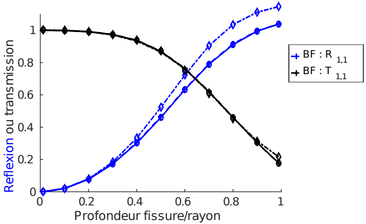
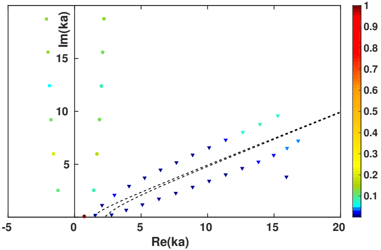
Low-frequency L(0,1) incident:

- $|z_i - z_{\text{ref}}| = 1a$, 13 leaky[a]
- $|z_i - z_{\text{ref}}| = 0.25a$, 13 leaky[a]
- $|z_i - z_{\text{ref}}| = 0.25a$, 13 leaky[a] **+27 PML modes**

[a] including forward and 'backward' modes





Spectrum: forward (circle), backward leaky (square), PML modes (triangle). Color: transmission coeff. ($h_n/a = 0.8$)

PML mode contribution can be significant in the near field:
a trade-off between FE box size and number of PML modes

Orthogonality... or not

Biorthogonality: $Q_{m,-n} = \frac{j\omega}{4}(\mathbf{F}_{-n}^{T}\mathbf{U}_m - \mathbf{U}_{-n}^{T}\mathbf{F}_m) = Q_{m,-m}\delta_{mn}$

Power non-orthogonality: the net power through cross-section Σ can be written as

$$\Pi_T = \sum_{m=-N}^{N} |\alpha_m|^2 \mathrm{Re}(P_{m,m}) + \sum_{m=-N}^{N}\sum_{n\neq m} \alpha_n^* \alpha_m P_{m,n} \quad \text{where } P_{m,n} = \frac{j\omega}{4}(\mathbf{F}_n^*\mathbf{U}_m - \mathbf{U}_n^*\mathbf{F}_m)$$

with $\mathrm{Re}(P_{m,m})$: power of nth mode, $P_{m,n}$: modal cross-power
In lossy problems, $P_{m,n} \neq Q_{m,-n} \rightarrow$ **'power non-orthogonality'**

Orthogonality... or not

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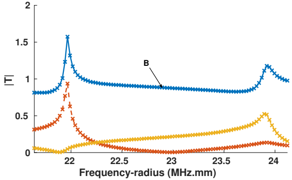
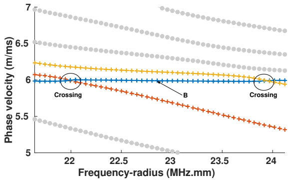
In lossy problems, $P_{m,n} \neq Q_{m,-n} \rightarrow$ **'power non-orthogonality'**

Consequence: individual power coefficients can be > 1, R and T can both increase...



blue: $|T_{12,12}|$, red: $|T_{13,12}|$, yellow: $|T_{14,12}|$ ($h_n/a = 0, 3$).



$\rightarrow$ high cross-power when $k_m \simeq k_n$

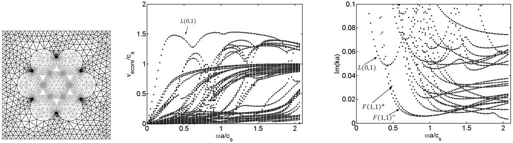
Conclusion

Using PML for the numerical modeling of open waveguides:

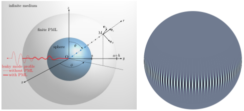
- leaky modes: revealed in a natural way (approximating the radiation continua)
- biorthogonality holds for any type of modes: including leaky! ~~(transverse growth)~~
- PML modes as a sum: physically meaningful (geometrically decaying field)...
- ... can be non-negligible in the 'deep' far field or in the 'close' near field
- open issues: completeness of expansion (convergence)?
  unicity of excitability ($Q_{m,-m}$ is a 'PMLized norm')?...

Main drawbacks in practice:

- PML parameters are user-defined
- computation time increases due to many PML modes for only a few leaky: what computational strategies?
  → an option to accelerate iterations over frequency? *Treyssède, "A model reduction method for fast finite element analysis of continuously symmetric waveguides", JSV 508 (2021)*



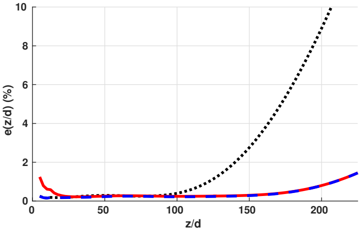
Modeling a buried seven wire strand: SAFE-PML mesh (left), energy velocity dispersion curves (middle) and attenuation (right)



Modeling a buried elastic sphere (gallery waves)

Influence of PML parameters on error vs. distance

Error vs. distance: $e(z) = \sqrt{\frac{\int |u_z^{\text{ref}}(z,\omega) - u_z^{\text{num}}(z,\omega)|^2 d\omega}{\int |u_z^{\text{ref}}(z,\omega)|^2 d\omega}}$



Relative error as a function of the propagation distance. $\hat{\gamma} = 4 + 4j$. $h = 4d$, $M = 50$; $h = 4d$, $M = 30$; $h = 3d$, $M = 50$.

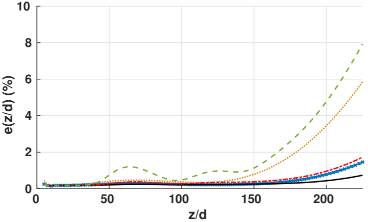
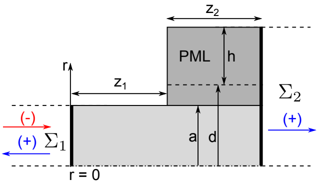


Figure: Influence of the argument for a complex thickness $25e^{j\theta}$: $\theta = 20^\circ$ (orange), $\theta = 30^\circ$ (blue), $\theta = 45^\circ$ (black), $\theta = 60^\circ$ (red), $\theta = 70^\circ$ (green).

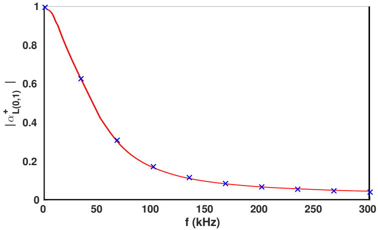
Reflection of L(0,1) mode by the junction of a steel bar with an infinite surrounding medium (epoxy)
Reference solution: Vogt *et al.*, JASA 2003 (FE element modelling + mode-matching)

Number of modes:

- M = 1 on Σ_1: L(0,1) guided mode
- M = 10 on Σ_2: L(0,1) leaky mode + 9 PML modes



Validation test case ($z_1 = z_2 = 0.25a$). PML parameters: $d = 1.05a$, $h = 2a$, $\hat{\gamma} = 4 + 4i$ (mean attenuation).



Red: reference results, blue: hybrid FE-SAFE.